Rational Loop Synthesis

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TU WIEN Informatics

based on joint work with

many great researchers

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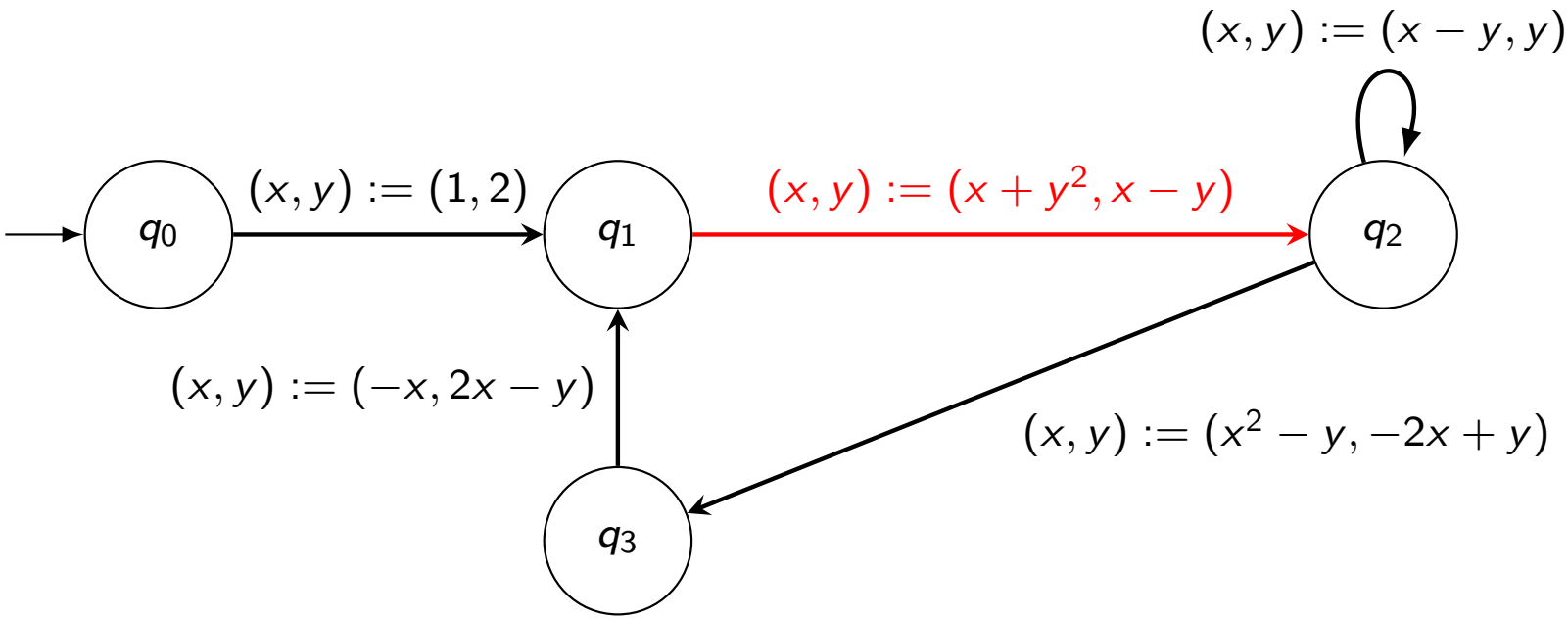
Loop Invariants and Algebraic Reasoning
Aarhus, Denmark
July 7th, 2025

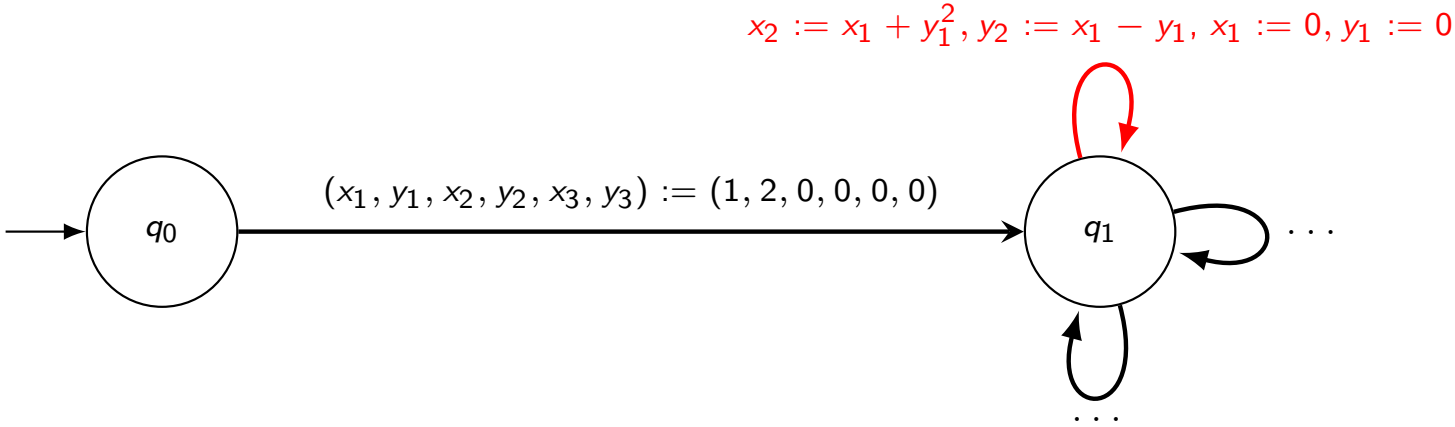
Polynomial Programs



ℓ non-initial locations, d variables

Goal: understand the reachable sets in $\mathbb{Q}^d$

Polynomial Loops



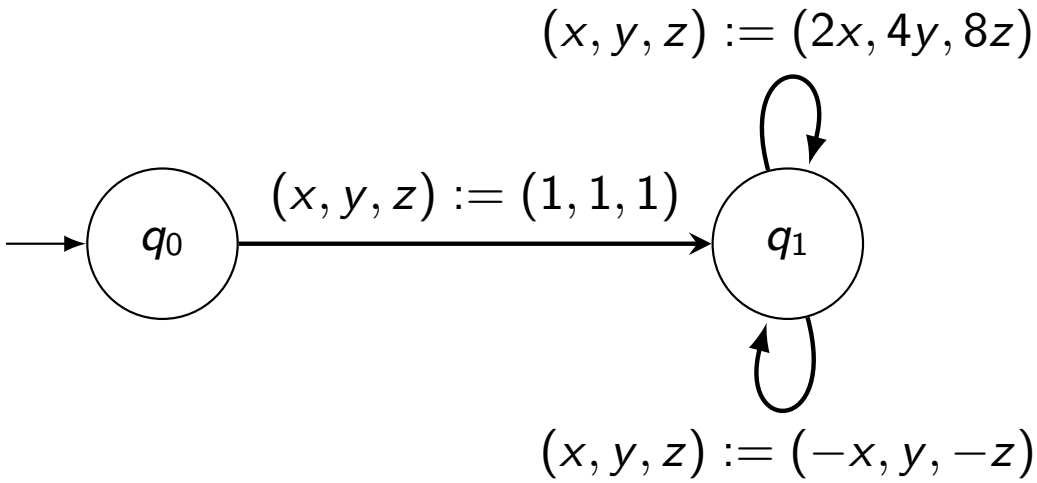
After reduction: $\ell \cdot d$ variables & same type of updates
Non-determinism, or "multi-path" loops

Focus on the set $S \subseteq \mathbb{Q}^{\ell d}$ of vectors reachable in q_1

Algebraic Invariants

an algebraic set

is a subset $V \subseteq \overline{\mathbb{Q}}^d$ defined by polynomial equalities.

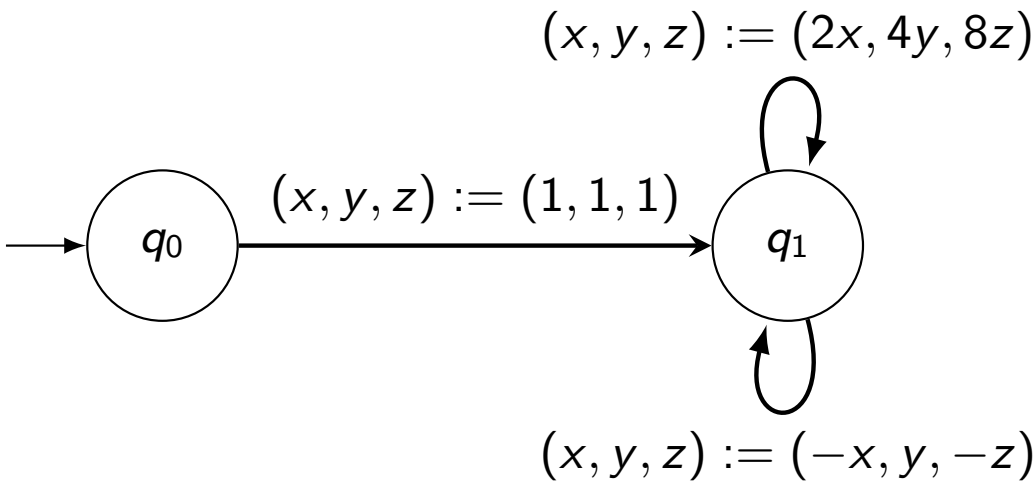


An algebraic set can overapproximate the reachable set S,
e.g., $S \subseteq V(\, x^2 - y \,)$. Then we call V an algebraic invariant of the loop.

Strongest Algebraic Invariant

Zariski closure $\overline{S}$ is the smallest algebraic set containing S.

smallest algebraic set $\leftrightarrow$ strongest algebraic invariant



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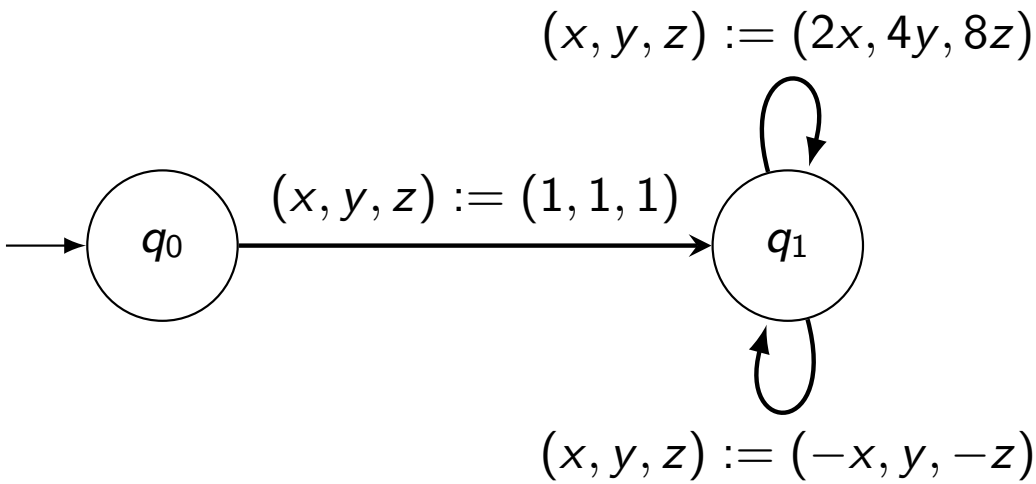

always a zero set of a finite collection of polynomials $\rightarrow$ finitary object

$$\overline{S} = V(\, x^2 - y, x^3 - z \,)$$

Beyond Linear Loops

Theorem [Kovács, V.] RAMiCS'23

The strongest algebraic invariant of a multi-path loop with polynomial updates of degree ≤ 2 is algorithmically **uncomputable**.

Reduction from the Boundedness Problem for Reset VASS.

Theorem [Müllner, Moosbrugger, Kovács] POPL'24

Computing the strongest algebraic invariant of a single-path polynomial loop is at least **as hard as Skolem problem**.

Open problem 1: Is it uncomputable?

Linear Loops

Thm. [Hrushovski, Ouaknine, Pouly, Worrell] LICS'18

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But what ARE the polynomials?

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Based on a polynomial-time procedure to compute multiplicative relations of M's eigenvalues.

But what ARE the polynomials?

Loop invariants: reverse engineering

$(x, y) := (0, 0)$
while $y < N$ **do**
  $x := x + 2y + 1$
  $y := y + 1$

??
holds before and
after each iteration

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$$(0,0), \quad (1,1), \quad (4,2), \quad \ldots$$

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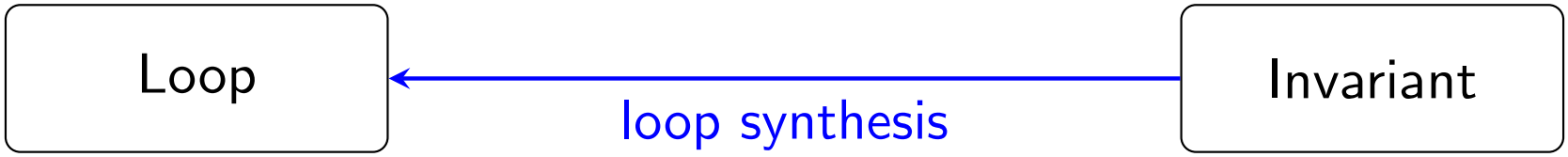

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For a polynomial invariant $p = 0$, synthesise a correct linear loop.

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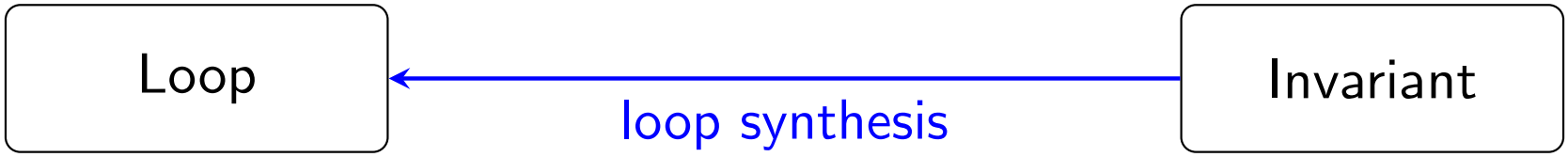

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For a polynomial invariant $p = 0$, synthesise a correct linear loop.

Decide whether such a loop exists.

Loop Synthesis problem

Given: a finite set of polynomials S
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- an update matrix $M \in \mathbb{Q}^{d \times d}$,
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Is there a loop?

$\boldsymbol{x} := \boldsymbol{s}$;

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such that

$$\mathcal{O} = \{\boldsymbol{s}, M\boldsymbol{s}, M^2\boldsymbol{s}, \dots\} \subseteq V(S) \quad \text{[weak]}$$

or

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☞ We always search for an infinite orbit $\mathcal{O}$.

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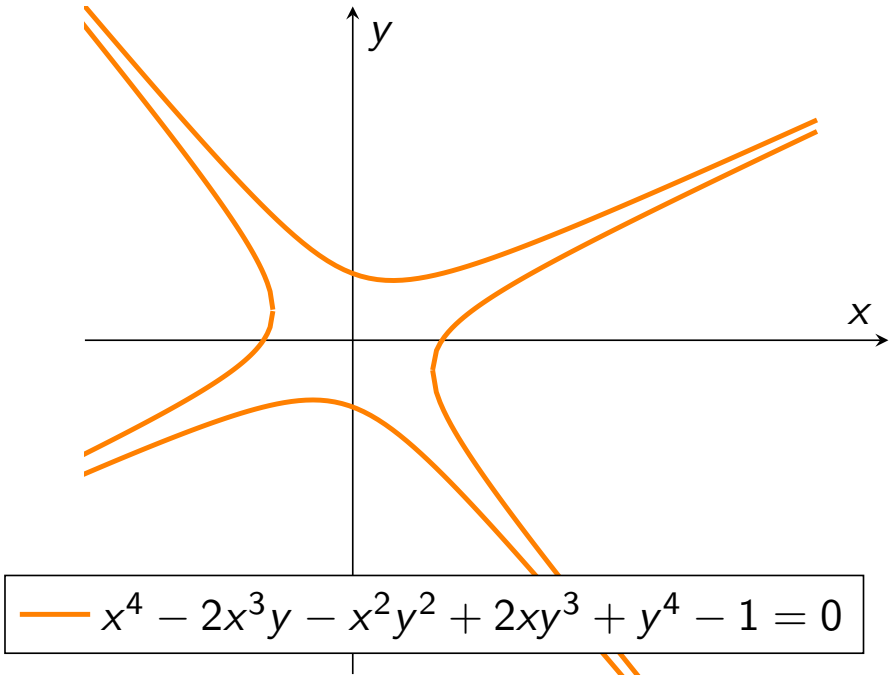
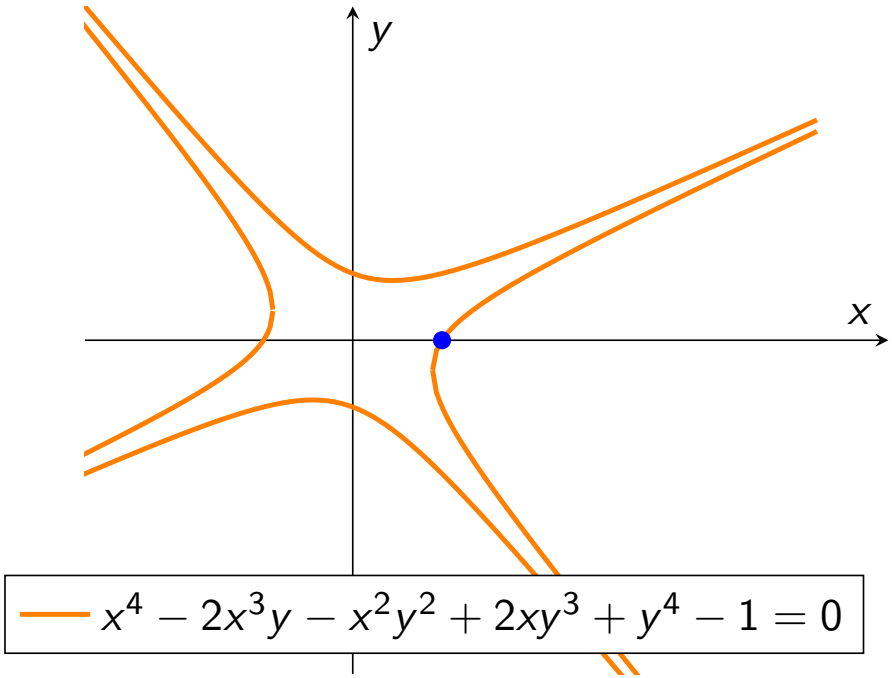
Loop synthesis: (algebra-)geometrically



sample rational points from a variety

☞ We always sample infinitely many points.

Loop synthesis: (algebra-)geometrically



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

sample rational points from a variety

$(x, y) \leftarrow (1, 0);$
while $*$ **do**
  $x \leftarrow x + y;$
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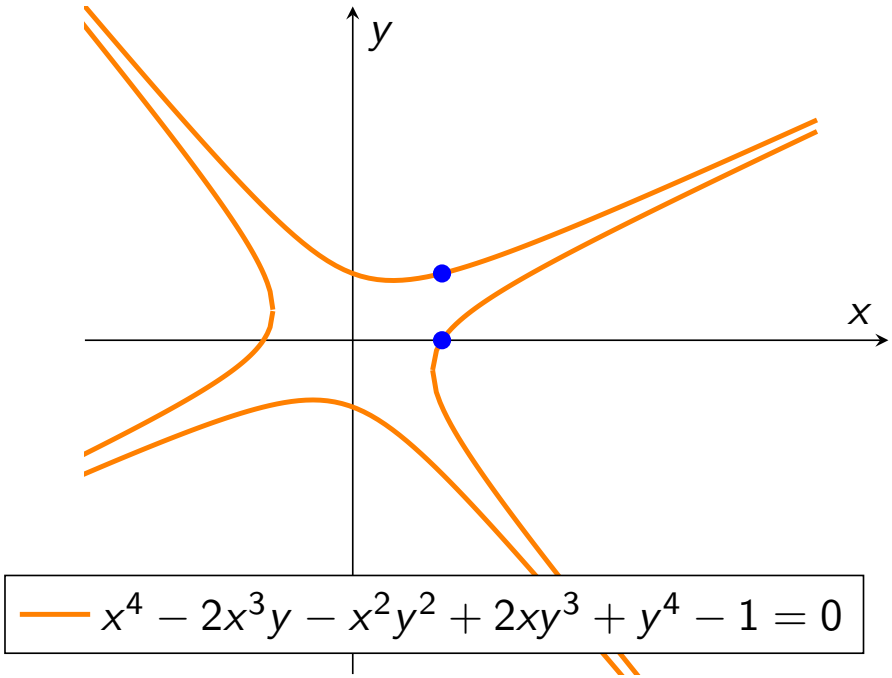

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ldots$$

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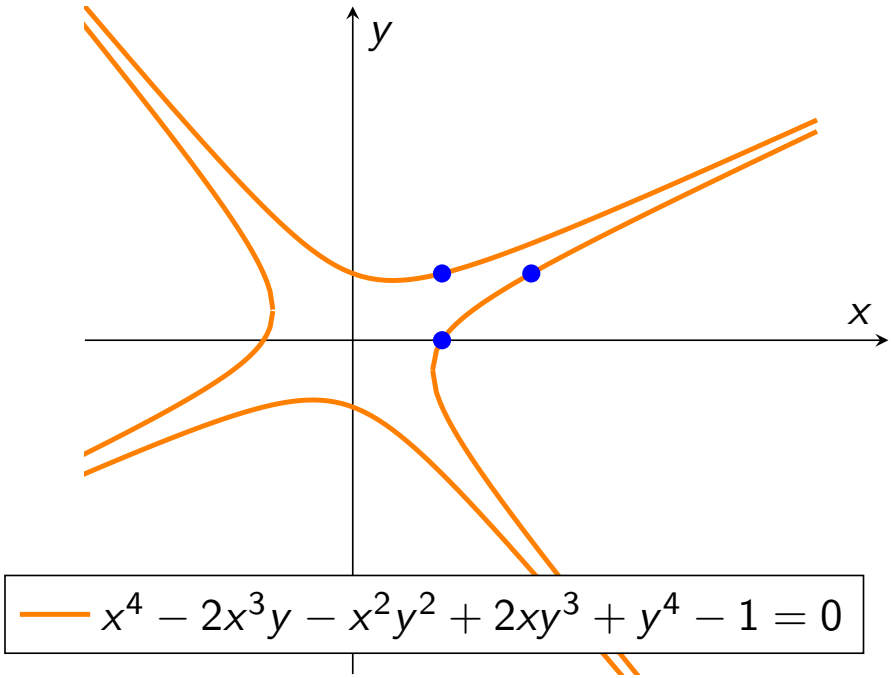

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ldots$$

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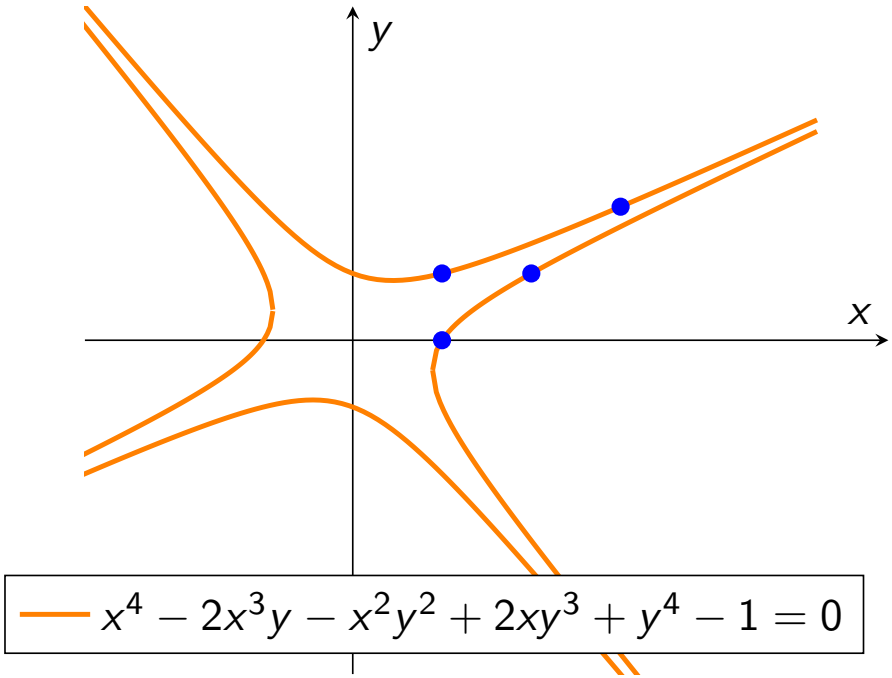

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \ldots$$

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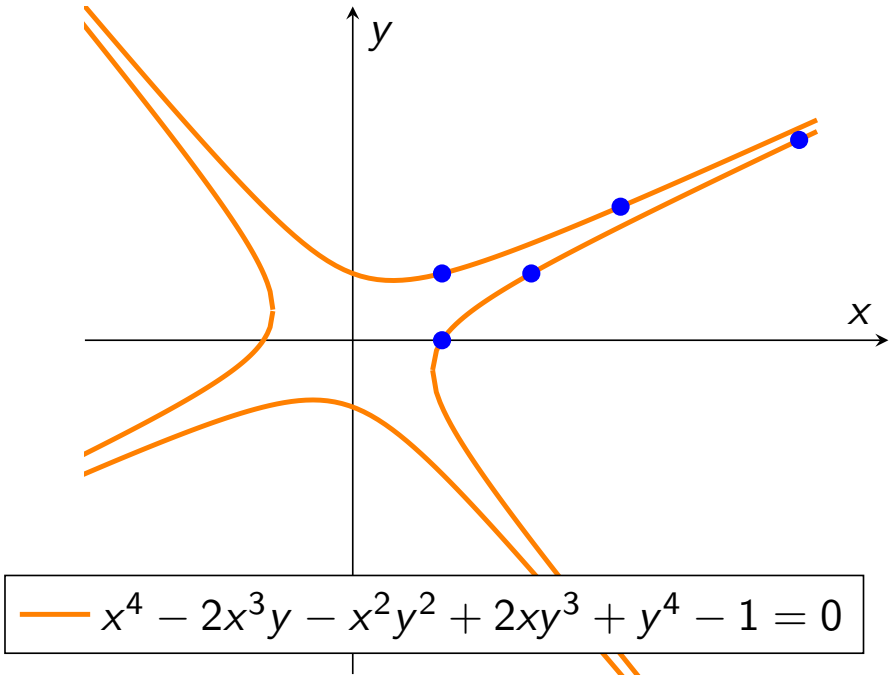

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \ldots$$

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Loop Synthesis is hard

Thm. [Ait El Manssour, Kenison, Shirmohammadi, V.] POPL'25

The weak synthesis problem over $\{\mathbb{Z}, \mathbb{Q}\}$ is as hard as Hilbert's 10th problem over $\{\mathbb{Z}, \mathbb{Q}\}$.

H10 over $R \in \{\mathbb{Z}, \mathbb{Q}\}$: solve a system of polynomial equations over R.

- undecidable for $R = \mathbb{Z}$,
- decidability open for $R = \mathbb{Q}$.

we encode an H10-instance $\{p_1, \ldots, p_k\}$ as finding a loop with algebraic invariant $V(p_1, \ldots, p_k, p)$

Pure difference binomials

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A pure difference binomial is a polynomial $p \in \mathbb{Q}[x_1, \ldots, x_d]$ of the form

$$p = x_1^{\alpha_1} \ldots x_d^{\alpha_d} - x_1^{\beta_1} \ldots x_d^{\beta_d},$$

where $\alpha_i, \beta_i \in \mathbb{N}$ for all $i = 1, \ldots, d$.

Examples in $\mathbb{Q}[x, y, z]$: $1 - xyz$, $xz - y^3$, $x^2y - z^3$, etc.

A Loop for PDBs

Loop Synthesis for PDB ideals

Input: pure difference binomials $p_1, \ldots, p_k$.

Output: A linear loop for which every $p \in I = \langle p_1, \ldots, p_k \rangle$ is invariant.

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Theorem [Kenison, Kovács, V.] ISSAC'23

Let $p_1, p_2, \ldots, p_k \in \mathbb{Q}[x_1, \ldots, x_d]$ be PDBs; let $I = \langle p_1, \ldots, p_k \rangle$.

- There exists a linear loop $\mathcal{L} = (M, \boldsymbol{s}) \in \mathbb{Q}^{d \times d} \times \mathbb{Q}^d$ s.t. $V(I)$ is its algebraic invariant.
- An effective procedure constructs $\mathcal{L}$.
- If $k < d$, then $\mathcal{L}$ has an infinite orbit.

Loops for systems of PDBs

We show how to combine multiple pure difference binomials.

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$(x, y, z) := (1, 1, 1);$
while $\star$ **do**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

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$$\begin{aligned} \langle x(n) \rangle_{n=0}^{\infty} &: \quad 1, 2, 4, \ldots \\ \langle y(n) \rangle_{n=0}^{\infty} &: \quad 1, 4, 16, \ldots \\ \langle z(n) \rangle_{n=0}^{\infty} &: \quad 1, 8, 64, \ldots \end{aligned}$$

For n-th terms, it holds: $x(n)^2 - y(n) = 0 \wedge x^3(n) - z(n) = 0$.

Multiplicative Relations

Assume we search for a diagonal matrix $M = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$.

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$$\left(\frac{x}{x_0}\right)^2 - \left(\frac{y}{y_0}\right) = (\lambda_1^n)^2 - \lambda_2^n = (2^n)^2 - (4^n) = 0 \textbf{ for all } n.$$

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A multiplicative relation

of $\lambda_1, \ldots, \lambda_d \in \overline{\mathbb{Q}}$ is a tuple $(v_1, \ldots, v_d) \in \mathbb{Z}^d$ s.t.

$$\lambda_1^{v_1} \ldots \lambda_d^{v_d} = 1.$$

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$(2, -1)$ is a mult. rel.

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Quadratic Invariants

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Thm. [Grunewald & Segal] (1981)

There is an algorithm to decide whether a quadratic equation in arbitrary number of variables has a rational solution.

$$\frac{3}{2} \cdot x^2 - 7 \cdot xy + \frac{4}{3} \cdot xz - z^2 + 31 \cdot y + z - 7 = 0$$

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$$\frac{3}{2} \cdot x^2 - 7 \cdot xy + \frac{4}{3} \cdot xz - z^2 + 31 \cdot y + z - 7 = 0$$

But: we still need to find a matrix M.

Consider an equation $Q(\boldsymbol{x}) = c$,

Q is a quadratic form (homogeneous).

$$x^2 + y^2 - z^2 = 0$$

$(x, y, z) := (3, 4, 5);$
while $\star$ **do**
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$(x, y, z) := (\mathbf{1}, \mathbf{1}, \mathbf{1})$;
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$(1, 1, 1) \rightarrow (1, 3, 3) \rightarrow (1, 5, 5) \rightarrow$
$\ldots$ a non-trivial loop!

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  $z := 2x - 2y + 3z;$

$(1, 1, 1) \to (1, 3, 3) \to (1, 5, 5) \to$
$\ldots$ a non-trivial loop!

How did we find this M?

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A loop for $Q = 2$:

$(x, y) := (\mathbf{3}, \mathbf{1})$;
while $\star$ **do**
  $x := 8x + 21y$;
  $y := 3x + 8y$;

A loop for $Q = -12$:

$(x, y) := (\mathbf{4}, \mathbf{2})$;
while $\star$ **do**
  $x := 8x + 21y$;
  $y := 3x + 8y$;

Theorem [Kenison, Kovács, Singh, V.] STACS'24

There exists a procedure that, given an equation $Q(x_1, \ldots, x_d) = c$, where Q is a quadratic form, decides whether **a non-trivial linear loop** satisfying it exists and, if so, synthesises a loop.

Arbitrary quadratic equations

Q is a quadratic form, L is a linear form of $\boldsymbol{x} = (x_1, \ldots, x_d)$.

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$$x^2 + y^2 - 3x - y = 0$$

$(x, y) := (2, -1)$
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$$\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 3/5x - 4/5y + 1 \\ 4/5x + 3/5y - 1 \end{pmatrix}$$

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NB: affine loops in d variables are linear loops in $d + 1$ variables:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} 3/5 & -4/5 & 1 \\ 4/5 & 3/5 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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Some polynomials need an additional variable.

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(alternative loop synthesis question:) given $I \subseteq \mathbb{Q}[x_1, \ldots, x_d]$, does there exist a loop $(M, \boldsymbol{s}) \in \mathbb{Q}^{k \times k} \times \mathbb{Q}^k$ for some $k \geq d$ s.t. $\mathcal{O} \subseteq V(I)$? Here $V(I) \subseteq \overline{\mathbb{Q}}^k$.

Open problem 2: is there an upper bound on k?

Bit-bounded version

Loop synthesis problem: is there a loop with algebraic invariant $V(S)$?
with additional input: integer B.
Search for $\langle M, \boldsymbol{s} \rangle$ with entries of bitsize $\leq B$, call them bit-bounded loops.

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- radical membership test to verify if $\overline{\mathcal{O}} \subseteq V(S)$ (**AM**)

More on bit-bounded

NP-hardness of the weak synthesis: reduce from 3SAT.
$\Phi = C_1 \wedge \cdots \wedge C_m$ with m clauses and d variables $y_1, \ldots, y_d$.

- add $x_i(1 - x_i)$ for each y_i to the polynomial collection S
- for each clause, say $C_i = y_1 \vee \neg y_2 \vee y_d$, add a polynomial p_i, here $(1 - x_1)x_2(1 - x_d)$.

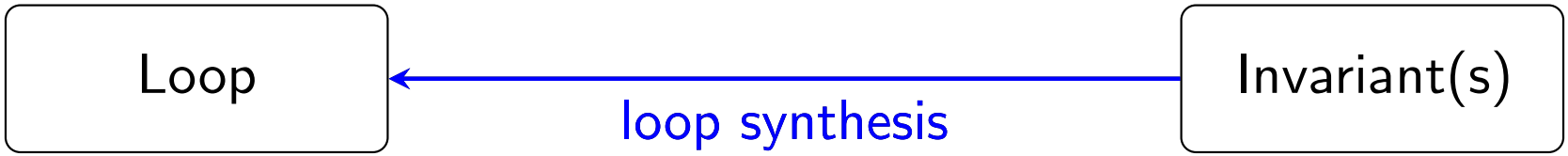
A non-trivial loop $M = \mathrm{diag}(1, \ldots, 1, 2, 2)$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d, 1, 1)$ has invariant $V(S) \Leftrightarrow \exists(\alpha_1, \ldots, \alpha_d)$, a sat. assignment

Open problem 3: Exact complexity of bit-bounded synthesis?

Open Problems

1. Is the strongest algebraic invariant of a single-path polynomial loop uncomputable?
2. Is there an upper bound on the minimal number of variables in a loop that has a given algebraic invariant?
3. What is the exact complexity of the bit-bounded synthesis problem?

Thank you ! Questions ?



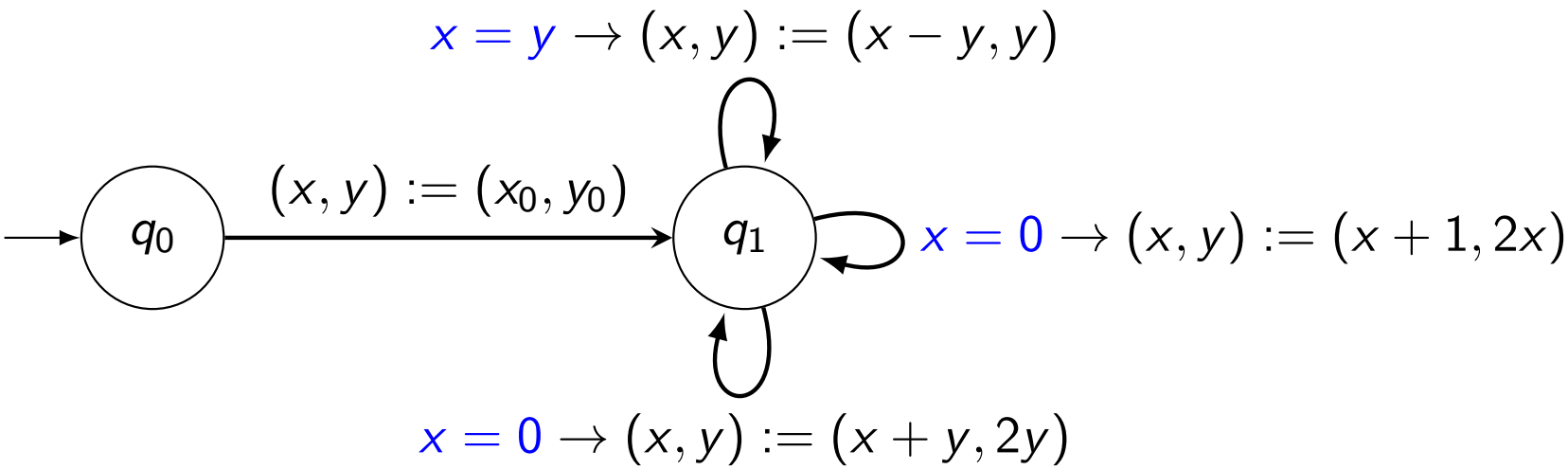
Procedures to synthesise simple (single-path) linear loops for

- invariant ideals generated by pure difference binomials $\rightarrow$ linear loops
- invariants defined by quadratic form equations $\rightarrow$ linear loops
- arbitrary quadratic equations $\rightarrow$ affine loops
- bit-bounded synthesis in **PH** and **NP**-hard

What Else is Unsolvable?

Proposition

Finding the strongest algebraic invariant of a multi-path affine loop with guarded affine updates is algorithmically **unsolvable**.



Multiple quadratic equations

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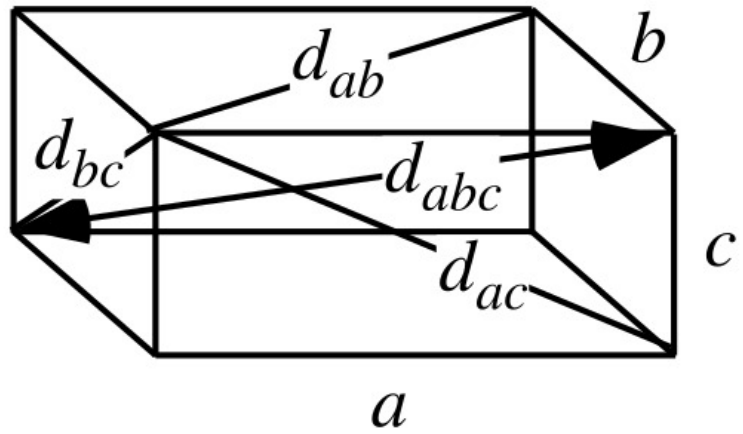
- A polynomial equation of any degree can be written as a system of linear and quadratic equations.
- **Perfect Euler brick**: a cuboid with edges and all four diagonals of integer length.

$$a^2 + b^2 = d_{ab}^2$$
$$b^2 + c^2 = d_{bc}^2$$
$$c^2 + a^2 = d_{ac}^2$$
$$a^2 + b^2 + c^2 = d_{abc}^2$$



Existence of a perfect Euler brick is **open**.

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- to diagonalise, get rid of xy : $x := x + \frac{1}{2}y, y := y \Rightarrow x^2 - \frac{5}{4}y^2 = 1$
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