Algebraic Tools for Computing Polynomial Loop Invariants

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Joint work with Erdenebayar Bayarmagnai and Rémi Prébet

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while true **do**

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end while

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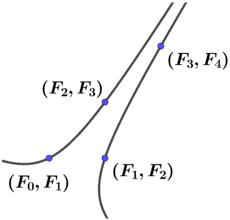

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Prior Works

- **Special classes of loops:**
  - Affine loops (Hrushovski, Ouaknine, Pouly, Worrell; LICS ’18)
  - P-solvable loops Kovács; TACAS ’08
  - Solvable loops (Rodriguez-Carbonell, Kapur; Symb. Comput. ’07)

- **Degree-bounded polynomial invariants:**
  - Synthesis for solvable loops (Amrollahi, Bartocci, Kenison, Kovács, Moosbrugger, Stankovic; Formal Methods Syst. Des. ’24)
  - Ideal-based reasoning (Cyphert Kincaid; ACM ’24)
  - Invariants from symbolic initialization (Müller-Olm, Seidl; Inf. Process. Lett. ’04)

(Semi)-algebraic loop

$(x_1, x_2, \dots, x_n) = (a_1, a_2, \dots, a_n)$

while $g_1 = \cdots = g_k = 0$ and $h_1 > 0, \dots, h_s > 0$ **do**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \overset{F}{\longleftarrow} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

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  - Generate all polynomial invariants of a specified algebraic form.
  - Algebraic formulation

Polynomial ideals

- Consider the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$.
- A subset $I \subseteq R$ is called a **polynomial ideal** if:
  - If $f, g \in I$, then $f + g \in I$ (closed under addition)
  - If $f \in I$ and $h \in R$, then $hf \in I$ (closed under multiplication in R).

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- For any ideal $I \subseteq R$, there exist polynomials $f_1, \ldots, f_r \in I$ such that

$$I = \langle f_1, \ldots, f_r \rangle.$$

- For any polynomial g in I there exists $h_1, \ldots, h_r$ in R such that

$$g = h_1 f_1 + \cdots + h_r f_r$$

Algebraic varieties

- Let $S = \{f_1, f_2, \ldots, f_s\} \subseteq \mathbb{C}[x_1, x_2, \ldots, x_n]$. Define

$$V(S) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n : f_i(a_1, \ldots, a_n) = 0 \text{ for all } 1 \leq i \leq s\}.$$

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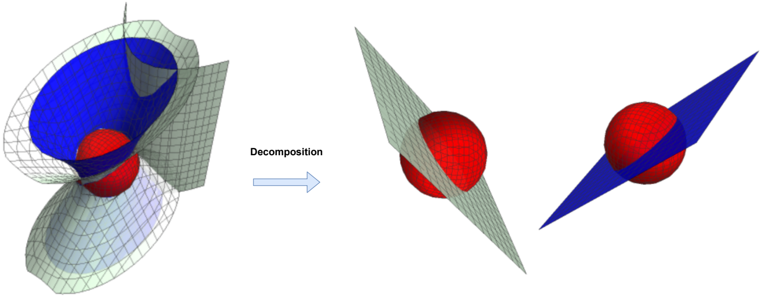
- If $\{f_1, \ldots, f_s\}$ and $\{g_1, \ldots, g_t\}$ generate the same ideal I, then:

$$V(f_1, \ldots, f_s) = V(g_1, \ldots, g_t)$$

- Equivalent polynomial systems



$\{x^2+y^2+z^2=1,\ 3x^2y^2-6x^2z+9x^2+y^4-y^2z^2-2y^2z+2y^2+2z^3-3z^2+2z-3=0,\ 5x^2-z^2+2y^2=2\}$ is decomposed to $\{x^2+y^2+z^2=1,\ x-z=0\}$ and $\{x^2+y^2+z^2=1,\ x+z=0\}$

Radical of ideals

Given an ideal $I \subseteq \mathbb{C}[x_1, x_2, \ldots, x_n]$, the **radical** of I is:

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$$I = (x_1^5x_5^5 + 5x_1^5x_5^4 - 10x_1^4x_3x_5^4 + 5x_1^4x_4x_5^4 + 10x_1^5x_5^3 - 40x_1^4x_3x_5^3 + 40x_1^3x_3^2x_5^3 + 20x_1^4x_4x_5^3 - 40x_1^3x_3x_4x_5^3 + 10x_1^3x_4^2x_5^3 + 10x_1^5x_5^2 - 60x_1^4x_3x_5^2 + 120x_1^3x_3^2x_5^2 - 80x_1^2x_3^3x_5^2 + 30x_1^4x_4x_5^2 - 120x_1^3x_3x_4x_5^2 + 120x_1^2x_3^2x_4x_5^2 + 30x_1^3x_4^2x_5^2 - 60x_1^2x_3x_4^2x_5^2 + 10x_1^2x_4^3x_5^2 + 5x_1^5x_5 - 40x_1^4x_3x_5 + 120x_1^3x_3^2x_5 - 160x_1^2x_3^3x_5 + 80x_1x_3^4x_5 + 20x_1^4x_4x_5 - 120x_1^3x_3x_4x_5 + 240x_1^2x_3^2x_4x_5 - 160x_1x_3^3x_4x_5 + 30x_1^3x_4^2x_5 - 120x_1^2x_3x_4^2x_5 + 120x_1x_3^2x_4^2x_5 + 20x_1^2x_4^3x_5 - 40x_1x_3x_4^3x_5 + 5x_1x_4^4x_5 + x_1^5 - 10x_1^4x_3 + 40x_1^3x_3^2 - 80x_1^2x_3^3 + 80x_1x_3^4 - 32x_3^5 + 5x_1^4x_4 - 40x_1^3x_3x_4 + 120x_1^2x_3^2x_4 - 160x_1x_3^3x_4 + 80x_3^4x_4 + 10x_1^3x_4^2 - 60x_1^2x_3x_4^2 + 120x_1x_3^2x_4^2 - 80x_3^3x_4^2 + 10x_1^2x_4^3 - 40x_1x_3x_4^3 + 40x_3^2x_4^3 + 5x_1x_4^4 - 10x_3x_4^4 + x_4^5,\ x_1^5 - 2x_3,\ x_3^3x_4^6 + 3x_1x_3^2x_4^4 + 3x_1^2x_3x_4^2 + x_1^3)$$

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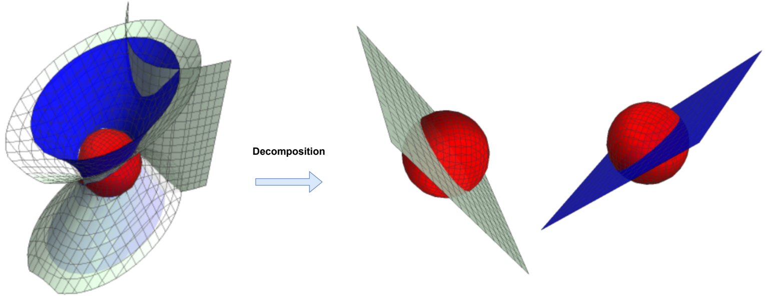
$$\text{rad}(I) = (x_1x_5 + x_1 - 2x_3 + x_4,\ x_1^5 - 2x_3,\ x_3x_4^2 + x_1)$$

- $V(I) = V(\text{rad}(I))$
- $I \subseteq \text{rad}(I)$

Description of polynomial invariant ideals

- Polynomial systems with the same solution set.



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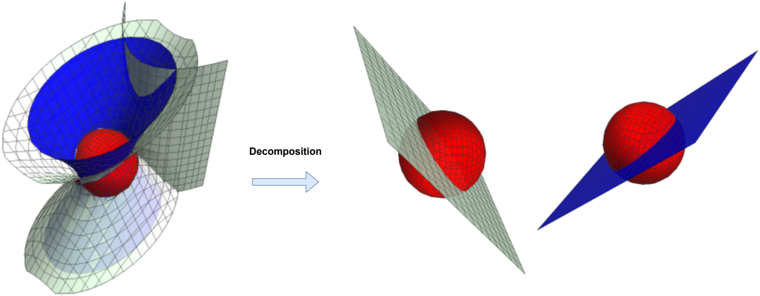

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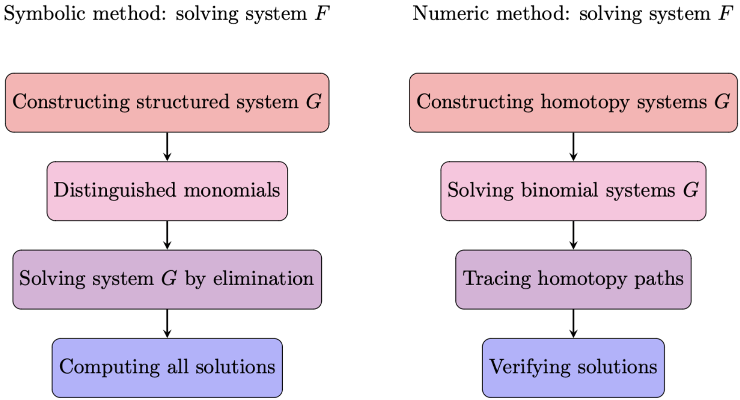
- Goals:
  - Find a **minimal** generating set for the ideal of polynomial invariants.
  - Decompose the associated variety into smaller ones
  - Generate equivalent generating sets which are easier to represent

Symbolic-numerical algebraic computation

- Approach: Degenerating a given system F to a more-structured system



- Main idea: Gröbner degenerations.

Algebraic formulation

g is a P.I. of $\mathcal{L}(a, F)$:

```
x ← a
while true do
    x ← F(x)
end while
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$\Longleftrightarrow$

$g(a) = 0$
$g \circ F(a) = 0$
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Definition

The invariant set of (F, g) is

$$S_{(F,g)} = \{x \in \mathbb{C}^n \mid \forall m \in \mathbb{Z}_{\geq 0} : g \circ F^{(m)}(x) = 0\}.$$

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Proposition

Let $a \in \mathbb{C}^n$. Then, g is a P.I. of $\mathcal{L}(a, F)$ if and only if $a \in S_{(F,g)}$.

Computing Invariant Sets

Theorem

Given a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ and a polynomial g, there exists an integer $N \in \mathbb{N}$ such that:

$$S_{(F,g)} = V(g) \cap V(g \circ F) \cap \cdots \cap V(g \circ F^{(N)}).$$

INVARIANTSETCOMPUTATION

Input: g and $F = (f_1, \ldots, f_n)$ in $\mathbb{Q}[x_1, \ldots, x_n]$
Output: A finite set of polynomials whose common zero set is $S_{(F,g)}$

$S \leftarrow \{g\}$
$\widetilde{g} \leftarrow g \circ F$
while $V(S) \neq V(S \cup \{\widetilde{g}\})$ **do**
    $S \leftarrow S \cup \{\widetilde{g}\}$
    $\widetilde{g} \leftarrow \widetilde{g} \circ F$
end while
return S

Example

$(\mathtt{x}_1, \mathtt{x}_2) \leftarrow (a_1, a_2)$
while true **do**
$$\begin{pmatrix} \mathtt{x}_1 \\ \mathtt{x}_2 \end{pmatrix} \leftarrow \begin{pmatrix} 10 & -8 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} \mathtt{x}_1 \\ \mathtt{x}_2 \end{pmatrix}$$
end while

- Consider $g = x_1^2 - x_1 x_2 + 9x_1^3 - 24x_1^2 x_2 + 16x_1 x_2^2$

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- $g \circ F = 360x_1^3 - 1248x_1^2 x_2 + 40x_1^2 + 1408x_1 x_2^2 - 72x_1 x_2 - 512x_2^3 + 32x_2^2$

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- Gröbner basis computation shows $V(g) \neq V(g, g \circ F)$.
- $g \circ F^2 = 7488x_1^3 - 26880x_1^2 x_2 + 832x_1^2 + 31744x_1 x_2^2 - 1600x_1 x_2 - 12288x_2^3 + 768x_2^2$
- This time, $V(g, g \circ F) = V(g, g \circ F, g \circ F^2)$.

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- This time, $V(g, g \circ F) = V(g, g \circ F, g \circ F^2)$.

Conclusion: g is a P.I. for $\mathcal{L}((a_1, a_2), F)$ if and only if $(a_1, a_2) \in V(g, g \circ F)$.

Polynomial invariants of degree d

$g = \sum_{|\alpha_i| \leq d} b_i x^{\alpha_i} \in \mathbb{C}[x]$ is a P.I.

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$h = \sum_{|\alpha_i| \leq d} y_i x^{\alpha_i} \in \mathbb{C}[x, y]$ is a P.I.

$(\mathtt{x}, \mathtt{y}) \leftarrow (a, b)$
while true **do**
  $(\mathtt{x}, \mathtt{y}) \leftarrow G(\mathtt{x}, \mathtt{y}) = (F(\mathtt{x}), \mathtt{y})$
end while

- Let $I_{d,\mathcal{L}}$ denote the set of all polynomial invariants of degree $\leq d$.
- It forms a finite-dimensional vector space and can thus be uniquely characterized by a system of linear equations.

Polynomial invariants of degree d

$g = \sum_{\lvert\alpha_i\rvert \leq d} b_i x^{\alpha_i} \in \mathbb{C}[x]$ is a P.I.	$h = \sum_{\lvert\alpha_i\rvert \leq d} y_i x^{\alpha_i} \in \mathbb{C}[x, y]$ is a P.I.

$\mathtt{x} \leftarrow a$
while true **do**
    $\mathtt{x} \leftarrow F(\mathtt{x})$
end while
	$(\mathtt{x}, \mathtt{y}) \leftarrow (a, b)$
	while true **do**
	    $(\mathtt{x}, \mathtt{y}) \leftarrow G(\mathtt{x}, \mathtt{y}) = (F(\mathtt{x}), \mathtt{y})$
	end while

- Let $I_{d,\mathcal{L}}$ denote the set of all polynomial invariants of degree $\leq d$.
- It forms a finite-dimensional vector space and can thus be uniquely characterized by a system of linear equations.

Theorem (ISSAC 2024)

Let $F = (f_1, \ldots, f_n)$ be a sequence of polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$ and let $d \geq 1$. Then, there is an algorithm that computes a polynomial matrix A, s.t.

$$I_{d,\mathcal{L}} = \{ \sum_{\lvert\alpha_i\rvert \leq d} b_i x^{\alpha_i} \mid (b_1, \ldots, b_m) \in \ker A(\mathbf{a}) \}.$$

Example

$(\mathtt{x}_1, \mathtt{x}_2) \leftarrow (a_1, a_2)$
while true **do**
$$\begin{pmatrix} \mathtt{x}_1 \\ \mathtt{x}_2 \end{pmatrix} \leftarrow \begin{pmatrix} 10 & -8 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} \mathtt{x}_1 \\ \mathtt{x}_2 \end{pmatrix}$$
end while

- **Goal:** Compute all polynomial invariants of degree ≤ 2.
- Consider $F = (10x_1 - 8x_2, \quad 6x_1 - 4x_2, \quad y_1, \ldots, y_6)$
- The general polynomial: $g = y_1 + y_2x_1 + y_3x_2 + y_4x_1^2 + y_5x_1x_2 + y_6x_2^2$
- The algorithm returns a matrix $M(x_1, x_2)$ such that

$$M(x_1, x_2) \cdot (y_1 \ y_2 \ \cdots \ y_6)^T = 0$$

encodes the polynomial invariants.

Output matrix and basis cases

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3x_1 - 4x_2 & 3x_1 - 4x_2 & 0 & 0 & 0 \\ 0 & 64x_2 & 112x_2 - 48x_1 & 48x_2^2 & 84x_2^2 - 36x_1x_2 & 27x_1^2 - 126x_1x_2 + 147x_2^2 \\ 0 & 32x_2 & 56x_2 - 24x_1 & 24x_1x_2 & -9x_1^2 + 21x_1x_2 + 12x_2^2 & -18x_1x_2 + 42x_2^2 \\ 0 & 4x_2 & 7x_2 - 3x_1 & 3x_1^2 & 3x_1x_2 & 3x_2^2 \end{bmatrix}$$

Output matrix and basis cases

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3x_1 - 4x_2 & 3x_1 - 4x_2 & 0 & 0 & 0 \\ 0 & 64x_2 & 112x_2 - 48x_1 & 48x_2^2 & 84x_2^2 - 36x_1x_2 & 27x_1^2 - 126x_1x_2 + 147x_2^2 \\ 0 & 32x_2 & 56x_2 - 24x_1 & 24x_1x_2 & -9x_1^2 + 21x_1x_2 + 12x_2^2 & -18x_1x_2 + 42x_2^2 \\ 0 & 4x_2 & 7x_2 - 3x_1 & 3x_1^2 & 3x_1x_2 & 3x_2^2 \end{bmatrix}$$

- We compute an explicit basis for the vector space $I_{2,\mathcal{L}}$ by computing the kernel of the above matrix, depending on the initial values (a_1, a_2).

Output matrix and basis cases

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3x_1-4x_2 & 3x_1-4x_2 & 0 & 0 & 0 \\ 0 & 64x_2 & 112x_2-48x_1 & 48x_2^2 & 84x_2^2-36x_1x_2 & 27x_1^2-126x_1x_2+147x_2^2 \\ 0 & 32x_2 & 56x_2-24x_1 & 24x_1x_2 & -9x_1^2+21x_1x_2+12x_2^2 & -18x_1x_2+42x_2^2 \\ 0 & 4x_2 & 7x_2-3x_1 & 3x_1^2 & 3x_1x_2 & 3x_2^2 \end{bmatrix}$$

- We compute an explicit basis for the vector space $I_{2,\mathcal{L}}$ by computing the kernel of the above matrix, depending on the initial values (a_1, a_2).
- Performing Gaussian elimination on M leads to four cases:

Initial values	Basis of $I_{2,\mathcal{L}}$
$a_1 = a_2 = 0$	$\{x_1, x_2, x_1x_2, x_1^2, x_2^2\}$
$a_1 = a_2 \neq 0$	$\{x_1 - x_2, x_1^2 - x_1x_2, -x_1x_2 + x_2^2\}$
$a_1 = \frac{4}{3}a_2 \neq 0$	$\{3x_1 - 4x_2, -3x_1^2 + 16x_1x_2 - 16x_2^2, -3x_1x_2 + 4x_2^2\}$
$a_1 \neq \frac{4}{3}a_2$, $a_1 \neq a_2$	$\{(3a_1-4a_2)^2x_1 - (3a_1-4a_2)^2x_2 - 9(a_1-a_2)x_1^2 + 24(a_1-a_2)x_1x_2 - 16(a_1-a_2)x_2^2\}$

Experiments (comparison with Polar)

- Polar: Moosbrugger, Stankovic, Bartocci, Kovács OOPSLA2, 2022
- Polar can handle **probabilistic loops**, whereas ours is limited to **deterministic** ones.
- We compute **all** possible polynomial invariants up to a specified degree,
- Ours are **minimal** generating polynomials of degree 1 to 4. TL = Timeout (360 seconds).

Degree	1		2		3		4	
Benchmark	Ours	Polar	Ours	Polar	Ours	Polar	Ours	Polar
Fib1	0.014	0.2	0.046	0.32	0.17	0.68	1.31	1.58
Fib2	0.017	0.23	0.056	0.46	6.3	1.18	TL	3.69
Fib3	0.013	0.21	0.056	0.4	0.137	1.26	0.61	3.82
Nagata	0.026	0.25	0.07	0.55	0.15	1.21	0.35	2.84
Yagzhev9	0.12	0.43	TL	5.2	TL	131.5	TL	TL
Yagzhev11	0.095	0.45	2.7	6.83	241	359	TL	TL
Ex 9	0.016	0.28	0.06	0.64	0.19	2.38	0.55	11.5
Ex 10	0.02	0.51	0.07	1.7	0.16	16.21	0.75	TL
Squares	0.02	0.5	0.06	0.67	0.15	1.15	0.38	2.25

Degree	1		2		3		4	
Benchmark	d	Polar	d	Polar	d	Polar	d	Polar
Fib1	0	0	0	0	1	1	**4**	1
Fib2	0	0	0	0	1	1	TL	**1**
Fib3	0	0	0	0	1	1	**4**	1
Nagata	**1**	0	**5**	1	**13**	1	**26**	2
Yagzhev9	**3**	0	TL	**3**	TL	**3**	TL	TL
Yagzhev11	0	0	0	0	TL	**1**	TL	TL
Ex 9	0	0	0	0	**3**	1	**11**	1
Ex 10	0	0	**2**	0	**8**	0	**19**	0
Squares	**1**	0	**5**	0	**13**	0	**26**	0

Summary

Summary

- Compute the set of initial values for which given polynomials are P.I.
- Compute all polynomial invariants up to a given degree:
  - For each possible initial value
  - For a fixed initial value
- Compute all polynomial invariants of a given form (fixed terms).
- Exploit the structure of polynomial systems for efficiency.
- Extend methods to loops with inequality guards (ISSAC 2024).

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- Bayarmagnai, Mohammadi, Prébet. ISSAC 2024

Thank you for your attention!