Finite Rational Matrix Semigroups have at most Exponential Size

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Workshop on Loop Invariants and Algebraic Reasoning

July 7 2025

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while true do
  if $x > 0$ **then**
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For programs with only linear updates $\mathbf{x} := A\mathbf{x} + \mathbf{b}$ and no conditional branching, *all* polynomial loop invariants can be automatically synthesized.

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$$\langle A_1, \dots, A_r \rangle := \{A_{i_1} \circ \dots \circ A_{i_n} \mid n \in \mathbb{N}, i_j \in \{1, \dots, r\}\}$$

Prior Work

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Theorem (Derksen, Jeandel, Koiran 2005)

Given invertible matrices $A_1, \ldots, A_r \in GL_n(\overline{\mathbb{Q}})$, the Zariski closure of $\langle A_1, \ldots, A_r \rangle$/the optimal polynomial invariants can be computed.

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Our goal is to reduce the complexity/generalize to semigroups.

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For the lower bound, reduce from DFA-intersection-emptiness. □

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If the group $G = \langle A_1, \ldots, A_r \rangle \subseteq GL_n(\mathbb{Q})$ is finite, then $|G| \leq 2^n \cdot n!$.

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Theorem (Our Main Tool)

Given a number field K *and semigroup* $S = \langle A_1, \ldots, A_r \rangle \subseteq K^{n \times n}$, *we can in PTIME compute an irreducible component decomposition of* S *over* K.

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Rest of this talk: Explains irreducible components + theorems

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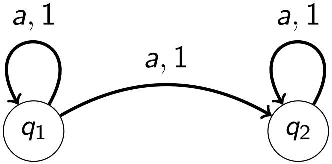

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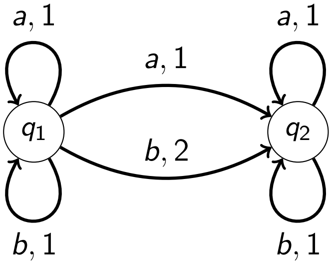

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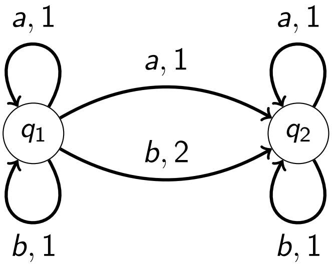

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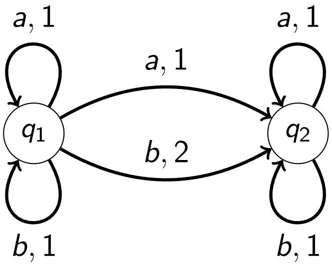

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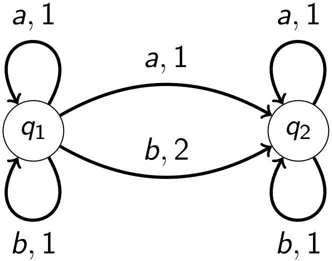

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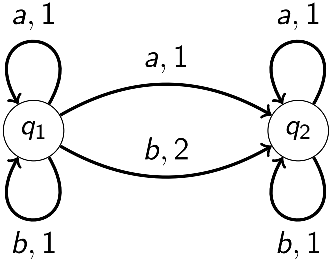

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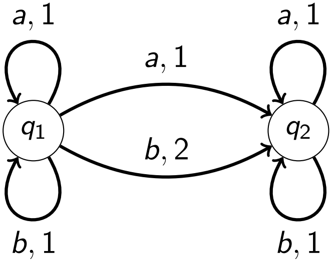

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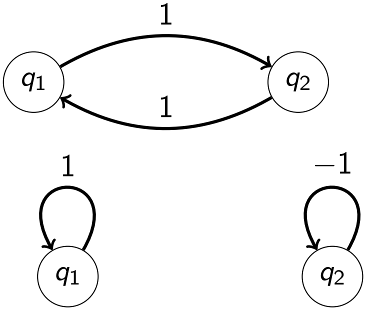

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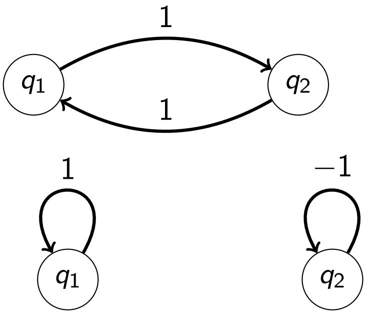

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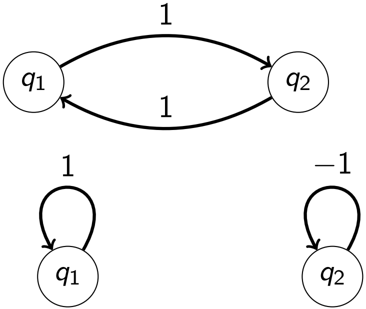

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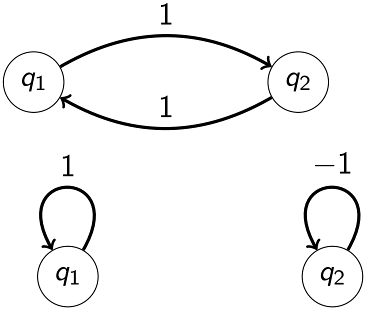

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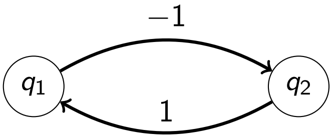

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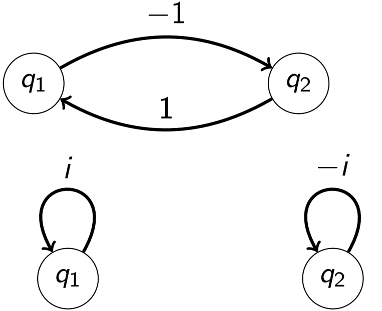
S is stably-strongly-connected if every base change PSP^{-1} is strongly-connected.

Mainly for non-unary alphabet, otherwise Jordan normal form.

Remark: Stably-strongly-connected depends on the number field.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$



Irreducible Semigroups

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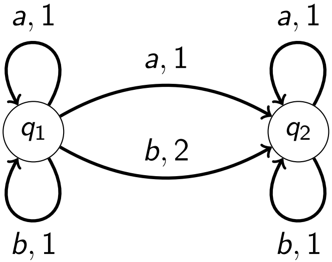

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S is stably-strongly-connected if every base change PSP^{-1} is strongly-connected.

S is irreducible if there is no vector space $\{0\} \neq V \subsetneq \mathbb{Q}^n$ s.t. $S \cdot V \subseteq V$.

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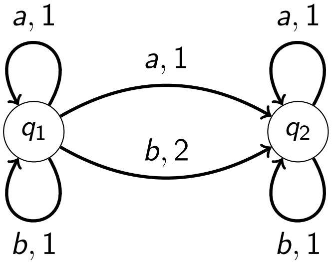

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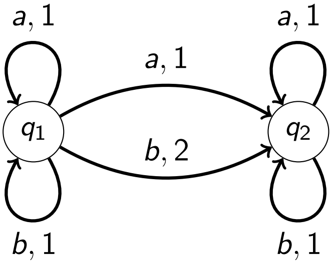

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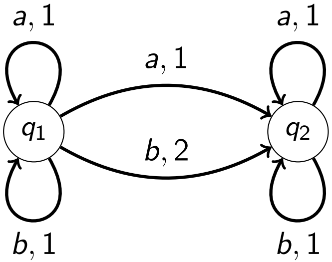

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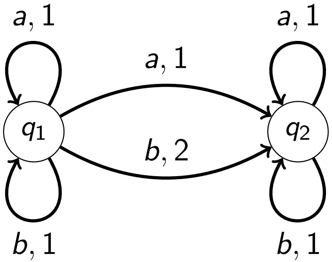

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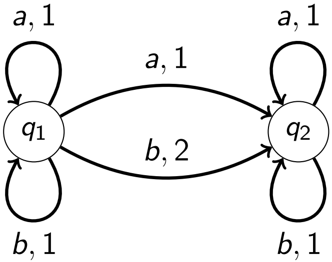

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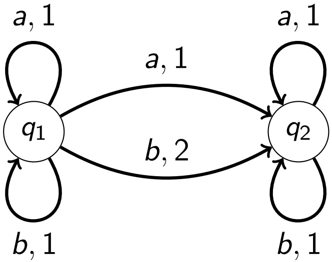

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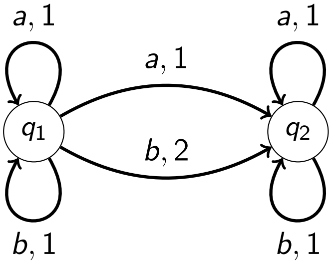

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$$

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An ICD always exists.

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Proof.

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If S is irreducible, then S is an ICD.
If S is reducible, then let V be s.t. $S \cdot V \subseteq V$. Decompose $S\mid_V$ and $S\mid_{V^\perp}$ recursively. $\square$

Reminder: Goal of This Talk

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Theorem (This Talk)

If the semigroup $S = \langle A_1, \ldots, A_r \rangle \subseteq \mathbb{Q}^{n \times n}$ is finite, then every matrix $A \in S$ has polynomial bitsize in terms of $A_1, \ldots, A_r$. (I.e. semigroups have $|S| \leq \exp.$)

Developed a tool which is applicable for general semigroups.

Theorem (Our Main Tool)

Given a number field K and semigroup $S = \langle A_1, \ldots, A_r \rangle \subseteq K^{n \times n}$, we can in PTIME compute an irreducible component decomposition of S over K.

Finite Semigroups: Bound on Bitsize

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In other words: We spent a large amount of this talk reducing to irreducible semigroups.

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T polysize $\Rightarrow$ T^{-1} polysize. $\square$

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Therefore $\lambda^m = \lambda^k$.
Hence either $\lambda = 0$ or $\lambda^{m-k} = 1$.

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Finite Semigroups: Bound on Bitsize

Theorem

In an irreducible finite semigroup $S = \langle A_1, \ldots, A_r \rangle \subseteq \mathbb{Q}^{n \times n}$, every matrix $A \in S$ has polynomial bitsize in terms of $A_1, \ldots, A_r$.

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Every eigenvalue λ of $A \in S$ fulfills $\lambda = 0$ or $\lambda^q = 1$ for some $q \in \mathbb{N}$.

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