Universal Skolem Sets

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(joint work with Florian Luca, James Maynard, Armand Noubissie, James Worrell)

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HOT MALE
STRIPPERS

Joël Ouaknine is with James Worrell and Amaury Pouly at Mathematical Institute, University of Oxford.

October 25, 2017 · Oxford ·

a mathematical storm is brewing

45

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Seek collaborations with people smarter than yourself

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The Skolem Problem

What do these sequences have in common?

- The Fibonacci numbers $\langle 0, 1, 1, 2, 3, 5, 8, \ldots \rangle$
- $\langle p(1), p(2), p(3), p(4), \ldots \rangle$
- $\langle \cos\theta, \cos 2\theta, \cos 3\theta, \cos 4\theta, \ldots \rangle$

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A **linear recurrence sequence (LRS)** is a sequence of integers $\langle u_0, u_1, u_2, \ldots \rangle$ such that there are constants $a_1, \ldots, a_k$ and
$\forall n \geq 0 : \quad u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n.$

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Problem SKOLEM (1934)

Instance: An LRS $\langle u_0, u_1, u_2, \ldots \rangle$
Question: Does $\exists n \geq 0$ such that $u_n = 0$?

Quick Quiz: two ‘simple’ problems

- Given two automata A and B, is there some ‘word-length’ n such that A and B accept exactly *the same words* of length n?
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Some other application areas

The Skolem Problem (and related questions) arise in many other areas (often in hardness results), e.g.:

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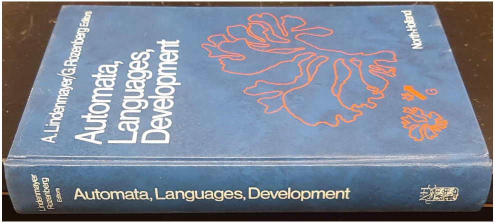
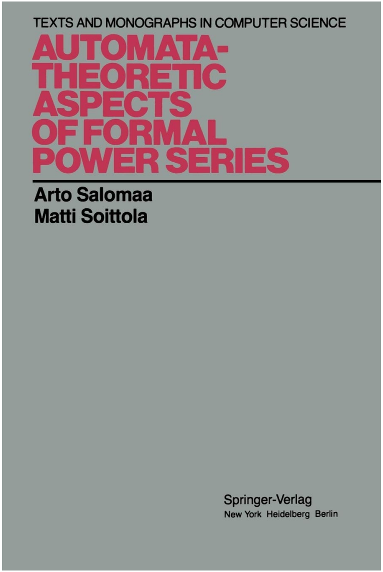
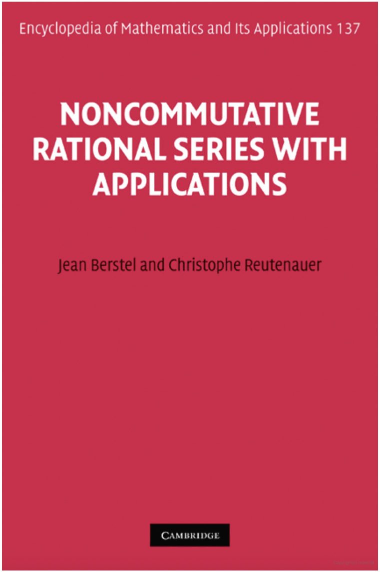
- theoretical biology (analysis of L-systems)
- software verification / program analysis
- dynamical systems
- differential privacy
- (weighted) automata and games
- analysis of stochastic systems
- control theory
- quantum computing
- statistical physics
- formal power series
- combinatorics
- ...

L-Systems (after Aristid Lindenmayer, late 1960s)



Automata and power series





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Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros $\{n \in \mathbb{N} : u_n = 0\}$ *of a non-degenerate LRS* $\langle u_0, u_1, u_2, \ldots \rangle$ *is finite.*

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- Decidability of the Skolem Problem is equivalent to being able to compute the finite set of zeros of any given non-degenerate LRS
- Unfortunately, all known proofs of the Skolem-Mahler-Lech Theorem make use of *non-constructive* p-adic techniques

Exponential-polynomial closed forms for LRS

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Then one has the exponential-polynomial closed form

$$u_n = \sum_{j=1}^{m} Q_j(n) \lambda_j^n$$

where the Q_j are polynomials with (complex) algebraic-number coefficients.

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- Simple LRS correspond precisely to **diagonalisable** matrices

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- Note: even for *simple* LRS, decidability at order 5 is not known!

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- Many, many results subject to P$\neq$NP, or ETH, etc...

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- Try our online tool SKOLEM !
  `https://skolem.mpi-sws.org/`

skolem.mpi-sws.org

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SKOLEM: Solves the Skolem Problem for simple integer LRS

System Explanation Show/Hide

- On the first line write the coefficients of the recurrence relation, separated by spaces.
- On the second line write an equal number of space-separated initial values.
- The LRS must be simple, non-degenerate, and not the zero LRS.
- The tool will output all zeros (at both positive and negative indices), along with a completeness certificate.

Input Format

$a_1\ a_2\ \dots\ a_k$
$u_0\ u_1\ \dots\ u_{k-1}$

where:
$u_{n+k} = a_1 \cdot u_{n+k-1} + a_2 \cdot u_{n+k-2} + \dots + a_k \cdot u_n$

Input area

Auto-fill examples: Show/Hide

Zero LRS | Degenerate LRS | Non-simple LRS | Trivial | Fibonacci | Tribonacci | Berstel sequence [1] | Order 5 [3] | Order 6 [3] | Reversible order 8 [3]

Manual input:

```
6 -25 66 -120 150 -89 18 -1
0 0 -48 -120 0 520 624 -2016
```

- Always render full LRS (otherwise restricted to 400 characters)
- I solemnly swear the LRS is non-degenerate (skips degeneracy check, it will timeout or break if the LRS is degenerate!)
- Factor subcases (merges subcases into single linear set, sometimes requires higher modulo classes)
- Use GCD reduction (reduces initial values by GCD)
- Use fast identification of mod-m (requires GCD reduction) (may result in non-minimal mod-m argument)

Go | Clear | Stop

Output area

Zeros: 0, 1, 4

Zero at 0 in (0+ 1ℤ) hide/show

- p-adic non-zero in (0+ 136$\mathbb{Z}_{\neq 0}$)
- Zero at 1 in (1+ 136ℤ) hide/show
  - p-adic non-zero in (1+ 680$\mathbb{Z}_{\neq 0}$) ((0+ 5$\mathbb{Z}_{\neq 0}$) of parent)
  - Non-zero mod 3 in (137+ 680ℤ) ((1+ 5ℤ) of parent)
  - Non-zero mod 3 in (273+ 680ℤ) ((2+ 5ℤ) of parent)
  - Non-zero mod 9 in (409+ 680ℤ) ((3+ 5ℤ) of parent)
  - Non-zero mod 3 in (545+ 680ℤ) ((4+ 5ℤ) of parent)
- Non-zero mod 7 in (2+ 136ℤ)

```
=================
LRS: u_{n} =
-27161311617120974485866352055894634704015095508906419136363354546754097691
1} +
-50875717942553060846492761332069658239718750163652943951247535707239324495
2} +
-10206640015864118991519942651944720249221599840966743554793056867782008052
3} +
-14120956624060003103644967151812606672989015750648229312685175908046543759
4} +
19069558947732071036098426589409142237569423390915870196544610694372734670
5} +
```

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An infinite set $\mathcal{S} \subseteq \mathbb{N}$ is a **Universal Skolem Set** if there is an effective procedure that inputs a non-degenerate integer LRS $\langle u_n \rangle$ and outputs the set $\{n \in \mathcal{S} : u_n = 0\}$.

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- Decidability of the Skolem Problem is equivalent to proving that $\mathbb{N}$ is a Universal Skolem Set
- In fact, it would suffice to show the existence of a Universal Skolem Set containing *some* infinite arithmetic progression!

Universal Skolem Sets exist!

Theorem (Luca, O., Worrell, LICS 2021)

Define $f : \mathbb{N}_+ \to \mathbb{N}$ *by* $f(t) = \lfloor \sqrt{\log t} \rfloor$. *Write* $s_0 = 1$ *and, inductively, set* $s_t := t! + s_{f(t)}$ *for* $t \geq 1$.
Then $\mathcal{S} := \{s_t : t \in \mathbb{N}\}$ *is a Universal Skolem Set.*

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We have

$$\mathcal{S} = \{1, 1! + 1, 2! + 1, 3! + 1! + 1, 4! + 1! + 1, 5! + 1! + 1, \ldots\}$$

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$$\begin{aligned}\mathcal{S} &= \{1, 1! + 1, 2! + 1, 3! + 1! + 1, 4! + 1! + 1, 5! + 1! + 1, \ldots\} \\ &= \{1, 2, 3, 8, 26, 122, 722, 5042, 40322, 362882, 3628802, \ldots\}\end{aligned}$$

Skolem-Universality of $\mathcal{S}$

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Proposition

Given $\langle u_n \rangle$, and any prime p such that $p \nmid \Delta$, then for all $t, \ell \in \mathbb{N}$ with $t \geq p^d$, $u_{t!+\ell} \equiv u_\ell \pmod{p}$.

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So $u_{t!+\ell} = \sum_{j=1}^{m} Q_j(t!+\ell)\lambda_j^{t!+\ell} \equiv \sum_{j=1}^{m} Q_j(\ell)\lambda_j^{\ell} = u_\ell \pmod{p}$.

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Then for any $t \geq N$, $u_{s_t} \neq 0$.

How dense is $\mathcal{S}$?

Recall $\mathcal{S} = \{1, 2, 3, 8, 26, 122, 722, 5042, 40322, 362882, \ldots\}$

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Unfortunately, $\mathcal{S}$ has density zero:

$$|\mathcal{S} \cap \{1, \ldots, n\}| \approx \frac{\log n}{\log \log n}$$

Exponential Diophantine equations in multiple variables

Theorem (after Schlickewei and Schmidt, 2000)

There is an explicit upper bound on the number of 'non-overlapping' solutions of the equation

$$\sum_{j=1}^{m} Q_j(y)\alpha_j^x\lambda_j^y = 0$$

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This is in fact a deep generalisation of the Skolem-Mahler-Lech Theorem

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Calculations show we can obtain unconditional density at least $1/2$.

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Bateman–Horn Conjecture. *Let $f_1, f_2, \ldots, f_k \in \mathbb{Z}[x]$ be distinct irreducible polynomials with positive leading coefficients, and let*

$$Q(f_1, f_2, \ldots, f_k; x) = \#\{n \le x : f_1(n), f_2(n), \ldots, f_k(n) \text{ are prime}\}. \quad (3.6.1)$$

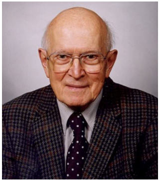
Suppose that $f = f_1 f_2 \cdots f_k$ does not vanish identically modulo any prime. Then

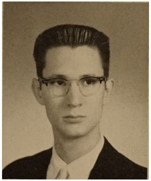
$$Q(f_1, f_2, \ldots, f_k; x) \sim \frac{C(f_1, f_2, \ldots, f_k)}{\prod_{i=1}^k \deg f_i} \int_2^x \frac{dt}{(\log t)^k}, \quad (3.6.2)$$

in which

$$C(f_1, f_2, \ldots, f_k) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\omega_f(p)}{p}\right) \quad (3.6.3)$$

and $\omega_f(p)$ is the number of solutions to $f(x) \equiv 0 \pmod p$.





The Bateman-Horn Conjecture

- It is a central, unifying, far-reaching statement about the distribution of prime numbers
- It implies many known results, such as the prime number theorem and the Green–Tao theorem, along with many famous conjectures, such the twin prime conjecture and Landau's conjecture
- It has been described as

  "ranking among the Riemann hypothesis and abc-conjecture as one of the most important and pivotal unproven conjectures in number theory"


BATEMAN-HORN
BATEMAN HOUSE
82-88 Hills Road
Vine FX
BenchSci
GTR
Intrasonics
University of Cambridge
Tucker Gardner
British Medical Group

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for $\mathfrak{p}$ a prime ideal above P and σ a Frobenius automorphism.

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Each representation (P, q, b) of n gives rise to a solution (q, b) of the **companion equation** (1) above.

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Each representation (P, q, b) of n gives rise to a solution (q, b) of the **companion equation** (1) above.

As the number of representations of n tends to infinity, but the number of solutions to the companion equation is explicitly bounded, this yields an effective upper bound on $n \in \mathcal{S}$ such that $u_n = 0$.

Outlook

Universal Skolem Sets are a radically new line of attack on the Skolem Problem

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- Is there a construction yielding a Universal Skolem Set containing *some* infinite arithmetic progression?
  - $\Rightarrow$ this would solve the Skolem Problem!
- Can these ideas be applied to other problems, such as Positivity or Ultimate Positivity, etc.?

"...on something like equal terms..."
[with apologies to G. H. Hardy]