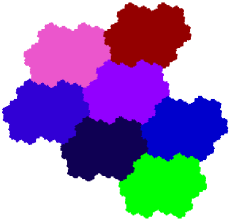
Recursive sequences and numeration

V. Berthé

IRIF-CNRS-Université Paris Cité



WORReLL'23

From linear recurrences to numeration systems

Start with your favourite linear recurrence sequence

Fibonacci recurrence

$$F_{n+1} = F_n + F_{n-1}$$

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Tribonacci recurrence

$$T_{n+2} = T_{n+1} + T_n + T_{n-1}$$

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Let β be the Tribonacci number, i.e., the real root of

$$X^3 - X^2 - X - 1 = 0$$

Let α be one of its conjugated complex roots. One has

$$\beta > 1, \quad |\alpha| < 1$$

The Tribonacci number β is a Pisot number

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Pisot-Vijayaraghavan number An algebraic integer is a Pisot number if its algebraic conjugates λ (except itself) satisfy

$$|\lambda| < 1$$

A brief overview

Linear recurrences $\rightsquigarrow$ numeration systems

$\rightsquigarrow$ numeration dynamical systems $\rightsquigarrow$ tilings

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- integers
- positive real numbers

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A linear recurrence allows the expansion of

- integers $\rightsquigarrow$ Odometer
- positive real numbers $\rightsquigarrow$ Beta-transformation

And the study of statistical properties of digits can be performed via their associated dynamical systems

- Finite expansions and periodic expansions
  - Average carry propagation for $+1$
  - Tilings and dynamical systems

Fibonacci numeration and carry prppagation

Fibonacci numeration scale (Zeckendorf representation)

$$F_0 = 1,\ F_1 = 2,\ F_{n+2} = F_{n+1} + F_n$$

$$N = \sum_{i=0}^{k} \varepsilon_i F_i, \quad \varepsilon_i \in \{0, 1\}$$

The set of greedy expansions of natural integers are words in $\{0, 1\}^*$ for which 11 is forbidden

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How does the carry propagate with respect to the successor function/$+1$?

$$11 \not\exists \rightsquigarrow 001$$

$$\begin{array}{rl} & N = F_1 + F_3 + F_6 + F_8 \\ & 010100101 \\ + & 1 \\ = & 000010101 \\ & \rightsquigarrow N+1 \end{array}$$

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$$F_0 = 1,\ F_1 = 2,\ F_{n+2} = F_{n+1} + F_n$$

$$\begin{array}{ccccccccccccc} \varepsilon & \to & 1 & \to & 10 & \to & 100 & \to & 101 & \to & 1000 & \to & 1010 & \to \\ & 1 & & 2 & & 3 & & 1 & & 4 & & 2 & & 5 \end{array}$$

	N	cp
ε	0	1
1	1	2
10	2	3
100	3	1
101	4	4

	N	cp
1000	5	1
1001	6	2
1010	7	5
10000	8	1
10001	9	2

	N	cp
10010	10	3
10100	11	1
10101	12	6
100000	13	1
100001	14	

Average carry propagation $= \dfrac{\tau}{\tau - 1} \simeq 2.618$

Base p

The successor map changes

- the least digit of every number
- plus another one every p numbers
- plus again another one every p^2 numbers, and so on...

The average carry propagation of the successor function, computed over the first N integers, tends to

$$1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{p}{p-1}$$

when N tends to infinity.

When dynamical systems enter into play

Let L be the set of representations of the integers
The average carry propagation CP_L is the limit, if it exists, of the mean of the carry propagation at the first N words of L

$$\mathrm{CP}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{cp}_L(n)$$

$$\mathrm{CP}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_L(\mathrm{Succ}_L^i(0))$$

This is an ergodic sum!

Ergodic theorem

We are given a dynamical system $(X, T, \mathcal{B}, \mu)$ with $T\colon X \to X$

- Average time values: one particle over the long term
- Average space values: all particles at a particular instant

Ergodicity

$$\mu(B) = \mu(T^{-1}B) \quad T\text{-invariance}$$

$$T^{-1}B = B \implies \mu(B) = 0 \text{ or } 1 \quad \text{ergodicity}$$

Ergodic theorem space mean= average mean

$$\text{If } f \in L_1(\mu), \quad \lim_N \frac{1}{N} \sum_{0 \le n < N} f(T^n x) = \int f \, d\mu \quad \text{a.e. } x$$

From the successor map to the odometer

- We have to turn the language L into a compact set.
- We work here with left infinite words since we use representations with most-significant digits on the left.
- One gets a compact set made of infinite words obtained by embedding in a dense way the representations of non-negative natural integers padded with 0's.
- We have to transform the successor function into a map of this compact set into itself.

$\rightsquigarrow$ odometers or adding machines [Grabner-Liardet-Tichy'95]

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Fibonacci odometer

$$\begin{array}{rr} & \cdots 101001010 \\ + & 1 \\ = & \cdots 101010000 \end{array}$$

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Base p

- The completion $\mathcal{K}_p$ is the ring $\mathbb{Z}_p$ of the p-adic integers.
- It is a topological group, and the odometer τ_p is just addition by one, and thus a group rotation

$$\rightsquigarrow \mathrm{CP}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \tau_p^i(0) = \frac{p}{p-1}$$

Exponential numeration scale

A numeration scale is an infinite sequence of positive integers and which plays the role of the sequence of the powers in an integer base.

It is a strictly increasing sequence of integers $G = (G_n)_{n \in \mathbb{N}}$ with $G_0 = 1$. The G-expansion of a natural integer N is the result of the greedy algorithm.

The numeration scale $G = (G_n)$ is exponential if

$$G_n \sim C\gamma^n$$

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Theorem [Barat-Grabner] If G is exponential, then $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic

Theorem [B.-Frougny-Rigo-Sakarovitch]
Let G be an exponential numeration. Then the average carry propagation exists. If L_G is prefix closed and right-extensible, then

$$\mathrm{CP}_G = \frac{\gamma}{\gamma - 1}$$

Numeration dynamics

Numeration dynamical systems are simple algorithms expressed in terms of dynamical systems that produce digits in classical representation systems

- Decimal expansions

$$T\colon [0,1] \to [0,1],\ x \mapsto 10x - [10x] = \{10x\}$$

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$$T\colon [0,1] \to [0,1],\ x \mapsto 10x - [10x] = \{10x\}$$

$$x_1 = T(x) = 10x - [10x] = 10x - a_1$$

$$x = \frac{a_1}{10} + \frac{x_1}{10}$$

$$x_2 = T(x_1) = T^2(x) \qquad a_2 = \lfloor 10\,T(x) \rfloor$$

$$x = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{x_2}{10^2} = \sum_{i=1}^{\infty} a_i 10^{-i}$$

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$$T\colon [0,1] \to [0,1],\ x \mapsto 10x - [10x] = \{10x\}$$

The map T produces the digits

$$a_n = \lfloor 10\,T^{n-1}(x) \rfloor$$

From numeration dynamics to symbolic dynamics

- Decimal expansion $T\colon [0,1] \to [0,1],\ x \mapsto \{10x\}$
- Beta-transformation $T\colon [0,1] \to [0,1],\ x \mapsto \{\beta x\}$
- Continued fractions $T\colon [0,1] \to [0,1],\ x \mapsto \{1/x\}$

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- Decimal expansion $T\colon [0,1] \to [0,1],\ x \mapsto \{10x\}$
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$$\beta > 1 \qquad x = \sum_{i=1}^{\infty} a_i \beta^{-i}$$

- Continued fractions $T\colon [0,1] \to [0,1],\ x \mapsto \{1/x\}$

$$x = \frac{1}{a_1 + x_1} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

β-numeration [Rényi'57, Parry'60]

Let $\beta > 1$. We consider the transformation

$$T_\beta(x)\colon [0,1] \to [0,1],\ x \mapsto \beta x - \lfloor \beta x \rfloor$$

Let $\mathcal{A}_\beta := \{0, 1, \cdots, \lceil \beta \rceil - 1\}$

Every real number $x \in [0,1]$ has an expansion of the form

$$x = \sum_{i \geq 1} u_i \beta^{-i}$$

$$\text{with} \quad u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor \in \mathcal{A}_\beta, \quad \forall i \geq 1$$

Admissible expansions and β-shift

Expansion of 1

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Let $d_\beta(1) = (t_n)_{n \geq 1}$. We set

$$\begin{cases} d^*_\beta(1) = d_\beta(1), \text{ if } d_\beta(1) \text{ is infinite} \\ d^*_\beta(1) = (t_1 \dots t_{m-1}(t_m - 1))^\omega, \text{ if } d_\beta(1) = t_1 \dots t_{m-1} t_m 0^\omega \ (t_m \neq 0) \end{cases}$$

Tribonacci numeration

$$\beta^3 = \beta^2 + \beta + 1, \ d_\beta(1) = 111, \ d^*_\beta(1) = 110110110 \cdots$$

Admissible expansions and β-shift

Expansion of 1

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will be used in the admissibility condition

- The set of β-expansions of real numbers in $[0, 1)$ is the set of sequences $(u_n)_{n \geq 1}$ with values in $\mathcal{A}_\beta$ such that

$$\forall k \geq 1,\ (u_n)_{n \geq k} <_{\text{lex}} d_\beta^*(1)$$

- The closure of this set is called the β-shift

$$\forall k \geq 1,\ (u_n)_{n \geq k} \leq_{\text{lex}} d_\beta^*(1)$$

β-shift= symbolic dynamical systems made of (bi-infinite) sequences of digits

Admissible expansions and β-shift

Expansion of 1

$$1 = \sum_{i \geq 1} t_i \beta^{-i}$$

will be used in the admissibility condition

We assume that β is Pisot.
Let $d_\beta(1)$ be the expansion of 1. Then, $d_\beta(1)$ is either finite or ultimately periodic.

The language of admissible words has a simple description as regular/sofic language

The finiteness property

- Fin (β) is the set of finite expansions
- Per (β) is the set of eventually periodic expansions
- Fin $(\beta) \subset$ Per (β)

Theorem [Bertrand-Schmidt] Let β be a Pisot number

$$\text{Per } (\beta) = \mathbb{Q}(\beta) \cap \mathbb{R}_+$$

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The (F) property [Frougny-Solomyak'92]

$$(F): \quad \text{Fin }(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}$$

Theorem [Frougny-Solomyak'92] The finiteness property (F) implies that β is a Pisot number

A sufficient condition [Frougny-Solomyak'92] Let β be the positive root of

$$X^m - a_1 X^{m-1} - \cdots - a_m \text{ where } a_1 \geq a_2 \geq a_m \cdots > 0$$

Then β is a Pisot number and β satisfies the (F) property

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Theorem [Akiyama] The origin is an exclusive inner point of the central tile/Rauzy fractal if and only if (F) holds

Toward Rauzy fractals

- Under which conditions is it possible to give a geometric representation of the β-shift as a map acting on a compact group?
- How to characterize real numbers in $[0, 1]$ that have a purely periodic expansion?
- How to characterize rational numbers in $[0, 1]$ that have a purely periodic expansion?

Periodic decimal expansions

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Let

$$T\colon \mathbb{Q}\cap[0,1]\to\mathbb{Q}\cap[0,1],\ x\mapsto 10x-[10x]=\{10x\}$$

Let $a/b\in[0,1]$ with b coprime with 10

$$T(a/b)=\frac{10a-[10\cdot a/b]\cdot\ b}{b}=\frac{10\cdot a \bmod b}{b}$$

- Denominator of $T^k(a/b)=b$
- Numerator of $T^k(a/b)$ belongs to $\{0,1,\cdots,b-1\}$

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We thus introduce

$$T'_b\colon x\mapsto 10\cdot x \bmod b$$

$$T'_b(a)\rightsquigarrow \text{numerator of } T(a/b)$$

We conclude by noticing that T'_b is onto and thus one-to-one since we work on a finite set

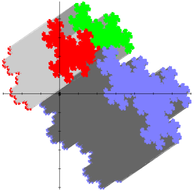
Purely periodic β-expansions

Beta-numeration Let $\beta > 1$. We expand real numbers $x \in [0, 1]$ as

$$x = \sum_{i \geq 1} u_i \beta^{-i}, \quad u_i \in \{0, \ldots, \lceil \beta \rceil - 1\}, \ \forall i \geq 1$$

A Lagrange theorem Theorem [K. Schmidt, A. Bertrand]
If β is a Pisot number, then x has an eventually periodic expansion iff $x \in \mathbb{Q}(\beta)$

A Galois theorem Theorem[Ito-Sano-Hui, B.-Siegel]
If β is a Pisot number, then x has a purely periodic expansion iff $(x, x') \in \widetilde{\mathcal{R}_\beta}$.



Central tiles for the β-shift [Rauzy, Thurston]

- We first need to give a meaning to the notion of conjugate x'
- We then are looking for a natural extension of the beta-transformation $T_\beta \colon x \mapsto \{\beta x\}$ which is not invertible
- We consider the past and future of orbits under T_β, i.e., the two-sided β-shift X_β

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To an element

$$(w,u) = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) \in X_\beta$$

we associate the formal power series

$$(\sum_{i\geq 0} w_i X^i, \sum_{i\geq 1} u_i X^{-i}) \quad \rightsquigarrow \quad (\sum_{i\geq 0} w_i \beta^i, \sum_{i\geq 1} u_i \beta^{-i})$$

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Meaning? Convergence?

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Take the conjugates of β (Pisot) and the prime ideals in the integer ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ of $\mathbb{Q}(\beta)$ that contain β

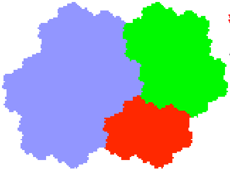
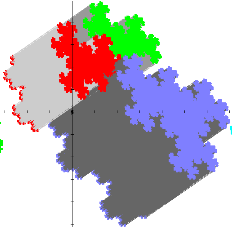
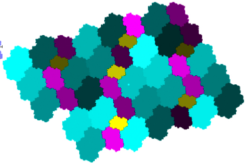
$$\widetilde{\mathcal{R}_\beta} \subset \mathbb{R}^{d-1} \times \prod_{p|N(\beta)} \mathbb{Q}_p^{d_p} \times \mathbb{R}$$

Tribonacci numeration

Let β stand for Tribonacci number (positive root of $X^3 - X^2 - X - 1 = 0$). Let α be one of its conjugate complex roots. One has $\beta > 1, |\alpha| < 1$ and $d_\beta(1) = 111$.

Let

$$\mathcal{R} = \{\sum_{i \geq 0} \varepsilon_i \alpha^i;\ \forall i,\ \varepsilon_i \in \{0,1\},\ \varepsilon_i \varepsilon_{i+1} \varepsilon_{i+2} = 0\}$$

Purely periodic expansions of rational numbers

We assume that β is a Pisot unit

$\gamma(\beta) = \sup\{0 < c < 1,$ every rational element in $[0, c[$ has a purely perio

$$\gamma(\beta) \geq A \text{ if diag } [0, A] \subset \mathcal{R}_\beta$$

Theorem [Akiyama] The origin is an exclusive inner point of the central tile/Rauzy fractal if and only if (F) holds

Theorem [Akiyama] If β is a Pisot unit satisfying the finiteness property, then $\gamma(\beta) > 0$

Theorem [Adamczewski-Frougny-Siegel-Steiner] The quantity $\gamma(\beta)$ is irrational when β is a cubic Pisot unit (not totally real case)

And now

- Carry propagation $\rightsquigarrow$ Interest of dynamical systems
- Finiteness property $\rightsquigarrow$ Decision problems, transcendence
- Canonical Number Systems, Shift Radix Systems

$$\sum_n M^n u_n$$

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