From Quantum Automata to Quaternions and Rational Pairing Functions

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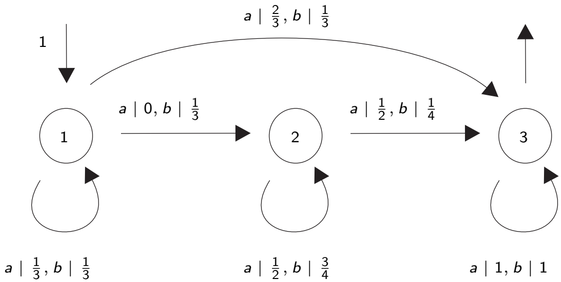
Probabilistic automata (Rabin 1963)



$$M_a = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix}, M_b = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{3}{4} & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{pmatrix}$$

Probabilistic automata

Definition

An n-state probabilistic automaton over Σ is a triplet

$$P = (\boldsymbol{y}, \{M_a \mid a \in \Sigma\}, \boldsymbol{x}),$$

where

- $\boldsymbol{x}$ is an initial stochastic (column) vector,
- $\boldsymbol{y} \in \{0, 1\}^n$ is the final (row) vector,
- each $M_a \in \mathbb{R}^{n \times n}$ is a stochastic matrix

Definition

The probability P associates to $w = a_1 \dots a_n \in \Sigma^*$ is given by

$$\mathbb{P}_P(w) = \boldsymbol{y} M_{a_n} \dots M_{a_1} \boldsymbol{x}.$$

Cut-point languages

Definition

For a stochastic automaton P and $\lambda \in [0,1]$ let

$$L_{\geq}(P,\lambda) = \{w \in \Sigma^* \mid \mathbb{P}_P(w) \geq \lambda\}$$

be a cut-point language and

$$L_{>}(P,\lambda) = \{w \in \Sigma^* \mid \mathbb{P}_P(w) > \lambda\}$$

a *strict* cut-point language.

Cut-point languages

Known properties

- Can define any regular language.
- Not necessary regular: $\{a^n b^n \mid n \in \mathbb{N}\}$ is a (strict) cut-point language (Turakainen 1969)
- If cutpoint is *isolated*, meaning that $(\exists \epsilon > 0)(\forall w)(\mathbb{P}_P(w) \notin (\lambda - \epsilon, \lambda + \epsilon))$ then regular (Rabin 1963)
- In the isolated case, at most exponential advantage over DFA size (Rabin 1963)

Measure-Once Quantum Automata (1997)

Definition

An n-state MO-QFA, aka Moore-Crutchfield QFA over Σ is a triplet

$$\mathcal{Q} = (P, \{U_a \mid a \in \Sigma\}, \boldsymbol{x}),$$

- Where $\boldsymbol{x} \in \mathbb{C}^n$ is the *initial vector* with the property $||\boldsymbol{x}|| = 1$,
- $P : \mathbb{C}^n \to \mathbb{C}^n$ is a projection,
- each $U_a \in \mathbb{C}^{n \times n}$ is a unitary matrix. $(U^*U = UU^* = I)$

Definition

The probability $\mathcal{Q}$ associates to $w = a_1 \dots a_n \in \Sigma^*$ is given by

$$\mathbb{P}_{\mathcal{Q}}(w) = ||PU_{a_n} \dots U_{a_1}\boldsymbol{x}||^2,$$

where $||\boldsymbol{x}||$ is the usual L_2-norm of $\boldsymbol{x}$.

Cut-point languages (MO-QFA)

Known properties

- For non-isolated cutpoint, simple examples of non-regular languages such as $\{w \mid |w|_a = |w|_b\}$.
- If cutpoint is isolated, then regular (Ablayev & al. 2000)
- But then only group languages can be recognized (Brodsky & Pippenger 2002)
- Example: Cannot recognize $\{a, b\}^* a$.

Decidability questions

Classical vs. Quantum

	$L_{\geq}(A, \lambda) = \emptyset$	$L_{>}(A, \lambda) = \emptyset$
PFA	Undecidable	Undecidable
QFA	Undecidable	Decidable

(Blondel & al. (binary alphabet, 47 states) 2003, Hirvensalo (25 & 21 states) 2007. Decidability assumes matrix entries from $\mathbb{Q}[i]$)

Injectivity problem for MO-QFA

Definition

Given a MO-QFA $\mathcal{Q}$, if the acceptance function of $\mathcal{Q}$ is injective, then we call $\mathcal{Q}$ injective.

Main Theorem (The injectivity problem)

Given a MO-QFA $\mathcal{Q}$, it is undecidable whether $\mathcal{Q}$ is injective.

Post Correspondence Problem

PCP

Given a collection of word pairs $(u_1, v_1), \ldots, (u_n, v_n)$ over an alphabet Δ, decide if there exists a nonempty index sequence $i_1 \ldots i_k$ so that

$$u_{i_1} u_{i_2} \ldots u_{i_k} = v_{i_1} v_{i_2} \ldots v_{i_k}?$$

Alternative formulation

Denote $w = i_1 i_2 \ldots i_k \in \Sigma^*$, where $\Sigma = \{1, 2, \ldots, n\}$ and define morphisms $h, g : \Sigma^* \to \Delta^*$ by $h(i_j) = u_{i_j}$, and $g(i_j) = v_{i_j}$. Does there exist a $w \in \Sigma^+$ so that

$$h(w) = g(w)?$$

https://webdocs.cs.ualberta.ca/~games/PCP/list.htm

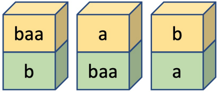


Figure: An easy case? (3 pairs)

Mixed Post Correspondence Problem

Mixed PCP

Given two morphisms $h, g : \Sigma^* \to \Delta^*$, decide if there exists a word $w = w_1 \dots w_n \in \Delta^+$ so that

$$h_1(w_1) \dots h_n(w_n) = g_1(w_1) \dots g_n(w_n),$$

where $h_i, g_i \in \{h, g\}$ and at least one $h_j \neq g_j$.

Theorem: Mixed PCP is undecidable

Injectivity problem for MO-QFA

Unitary embedding

- Let U_a and U_b be unitary matrices generating a free group. Then there is an embedding $w \to U_w$ from alphabet $\{a, b\}$ into the group $\langle U_a, U_b \rangle$.
- Let $h, g : \Sigma \to \{a, b\}^*$ be morphisms of a mixed PCP instance. Let also $e : \Sigma \to \{a, b\}^*$ be an embedding.
- For each $\sigma \in \Sigma$ define two unitary matrices $X_\sigma^h = U_{e(\sigma)} \oplus U_{h(\sigma)}$ and $X_\sigma^g = U_{e(\sigma)} \oplus U_{g(\sigma)}$, and let X be the union of those matrices.
- Define a QFA with matrices X and input alphabet $\Sigma \times \{g, h\}$.
- For an input word $w = (\sigma_1, f_1) \ldots (\sigma_n, f_n) = (u, v)$ we have $X_w = (U_{e(\sigma_1)} \oplus U_{f_1(\sigma_1)}) \ldots (U_{e(\sigma_n)} \oplus U_{f_n(\sigma_n)}) = U_{e(u)} \oplus U_{f_v(u)}$, where $f_v(u) = f_1(\sigma_1) \ldots f_n(\sigma_n)$. Hence
- $X_{w_1} = X_{w_2} \iff U_{e(u_1)} \oplus U_{f_{v_1}(u_1)} = U_{e(u_2)} \oplus U_{f_{v_2}(u_2)} \iff u_1 = u_2 = u$ and $f_{v_1}(u) = f_{v_2}(u)$.

Injectivity problem for MO-QFA

Notice that:

- Mixed PCP has a solution iff there are words $w_1 \neq w_2$ so that $X_{w_1} = X_{w_2}$
- The construction requires unitary embedding $w \to U_w$, $X_\sigma^f = U_{e(\sigma)} \oplus U_{f(\sigma)}$
- However, the mapping $X_w \to ||PX_w\mathbf{x}||^2$ may not be injective, meaning that the acceptance probability does not uniquely determine X_w.

Embedding to Binary Alphabet

Embedding of γ_1

Let $\Sigma_n = \{a_1, \ldots, a_n\}$ and $\Sigma_2 = \{a, b\}$. Then $\gamma_1 : \Sigma_n \to \Sigma_2^*$ is an embedding where $\gamma_1(a_k) = a^k b$

Extension

- Can be extended to $\gamma_1 : \Sigma_n^* \to \Sigma_2^*$ by $\gamma_1(w_1 w_2 \cdots w_k) = \gamma_1(w_1)\gamma_1(w_2 \cdots w_k)$.

Embedding to Quaternions

Embedding of γ_2

Let $\Sigma_2 = \{a, b\}$ and define $\gamma_2 : \Sigma_2 \to \mathbb{H}(\mathbb{Q})$ by $\gamma_2(a) = (\frac{3}{5}, \frac{4}{5}\mathbf{i}, 0, 0)$ and $\gamma_2(b) = (\frac{3}{5}, 0, \frac{4}{5}\mathbf{j}, 0)$ with $\gamma_2(\varepsilon) = I_4$. Note that $\{\gamma_2(a), \gamma_2(b)\}$ represent rotations about perpendicular axes by a rational angle

Theorem (Swierczkowski)

- If $\cos(\theta) \in \mathbb{Q}$ then the subgroup of $SO_3(\mathbb{R})$ generated by rotations of angle θ about two perpendicular axes is free iff $\cos(\theta) \neq 0, \pm\frac{1}{2}, \pm 1$.

Proposition

Thus $\langle \gamma_2(a), \gamma_2(b) \rangle$ is freely generated and γ_2 is an injective homomorphism

Embedding to Unitary Matrices

Embedding of $\gamma_3 : \mathbb{H}(\mathbb{Q}) \to \mathbb{Q}^{4\times 4}$

$$\gamma_3(\gamma_2(a)) = U_a = \frac{1}{5}\begin{pmatrix} 3 & 4 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -4 & 3 \end{pmatrix}, \quad \gamma_3(\gamma_2(b)) = U_b = \frac{1}{5}\begin{pmatrix} 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & -4 \\ -4 & 0 & 3 & 0 \\ 0 & 4 & 0 & 3 \end{pmatrix}$$

Denote $U_w = U_{w_1} \dots U_{w_n}$ and $R_1(U_w) = (\left|(U_w)_{11}\right|, \left|(U_w)_{12}\right|, \left|(U_w)_{13}\right|)$.

Theorem

- U_a and U_b generate a free group.
- If $R_1(U_u) = R_1(U_v)$, then $u = v$.
- Requires analysis of quaternion structure for $\gamma_2(a)$ and $\gamma_2(b)$.

Our injective homomorhism γ

Final embedding

By the previous observations $\gamma : \Sigma_k^* \to \mathbb{Q}^{4\times 4}$ is an injective homomorphism, where $\gamma(w) = \gamma_3(\gamma_2(\gamma_1(w)))$

Injectivity problem for MO-QFA

Observation

- Matrix $X_u^{f_v} = U_{e(u)} \oplus U_{f_v(u)}$ is fully determined by $R_1(U_{e(u)})$ and $R_1(U_{f(u)})$.
- Mixed PCP has a solution iff there is $(u_1, v_1) = w_1 \neq w_2 = (u_2, v_2)$ so that $R_1(U_{e(u_1)}) = R_1(U_{e(u_2)})$ (which implies $u_1 = u_2 = u$) and $R_1(U_{f_{v_1}(u)}) = R_1(U_{f_{v_2}(u)})$
- Mixed PCP has a solution iff there is $(u_1, v_1) = w_1 \neq w_2 = (u_2, v_2)$ so that

$$\begin{aligned} & (|X_{w_1}|_{11}, |X_{w_1}|_{12}, |X_{w_1}|_{13}, |X_{w_1}|_{55}, |X_{w_1}|_{56}, |X_{w_1}|_{57}) \\ = \; & (|X_{w_2}|_{11}, |X_{w_2}|_{12}, |X_{w_2}|_{13}, |X_{w_2}|_{55}, |X_{w_2}|_{56}, |X_{w_2}|_{57}) \end{aligned}$$

Tools

Lemma

a) There exist MO-QFA Q_0 and Q_1 so that $\mathbb{P}_{Q_0}(w) = 0$ and $\mathbb{P}_{Q_1}(w) = 1$ for each $w \in \Sigma^*$.

b) Given two MO-QFA's Q_1 and Q_2, complex numbers α and β so that $|\alpha|^2 + |\beta|^2 = 1$, there exists

  b.1) A MO-QFA Q so that $\mathbb{P}_Q(w) = \mathbb{P}_{Q_1}(w)\mathbb{P}_{Q_2}(w)$

  b.2) A MO-QFA Q so that $\mathbb{P}_Q(w) = |\alpha|^2 \mathbb{P}_{Q_1}(w) + |\beta|^2 \mathbb{P}_{Q_2}(w)$

Proof

a) Trivial b.1) Tensor product construction b.2) Direct sum construction

Observation

If $P = \text{diag}(0, \ldots, 1, \ldots, 0)$ (jth position) and $\boldsymbol{x} = (0, \ldots, 1, \ldots, 0)$ (ith position), then

$$||PU\boldsymbol{x}||^2 = |U_{ij}|^2 .$$

Reduction to Mixed PCP

- According to a previous observation, there is a MO-QFA which, on input $w = (u, v)$, produces output (acceptance probability) $|(X_w)_{ij}|^2 = \left|(U_{e(u)} \oplus U_{f_v(u)})_{ij}\right|^2$
- From the construction tools, it follows that there exists a MO-QFA producing output (acceptance probability)

$$|\lambda_1|^2 |(X_w)_{11}|^2 + |\lambda_2|^2 |(X_w)_{12}|^2 + |\lambda_3|^2 |(X_w)_{13}|^2 + |\kappa_1|^2 |(X_w)_{55}|^2 + |\kappa_2|^2 |(X_w)_{56}|^2 + |\kappa_3|^2 |(X_w)_{57}|^2 , \quad (1)$$

  where $\lambda_1, \lambda_2, \lambda_3, \kappa_1, \kappa_2, \kappa_3$ are complex numbers satisfying $|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2 + |\kappa_1|^2 + |\kappa_2|^2 + |\kappa_3|^2 = 1$
- Mixed PCP has a solution if and only if the same acceptance probability (1) can be obtained for at least two words $w_1 \neq w_2$ (Meaning that the automaton is ambiguous or not injective)

Undecidability

Conclusion

- For the final conclusion, we have to be sure that mapping

$$
\begin{aligned}
& (|X_{11}|, |X_{12}|, |X_{13}|, |X_{55}|, |X_{56}|, |X_{57}|) \\
\rightarrow \quad & |\lambda_1|^2 |X_{11}|^2 + |\lambda_2|^2 |X_{12}|^2 + |\lambda_3|^2 |X_{13}|^2 \\
+ \quad & |\kappa_1|^2 |X_{55}|^2 + |\kappa_2|^2 |X_{56}|^2 + |\kappa_3|^2 |X_{57}|^2
\end{aligned}
$$

  is injective.
- If now $|\lambda_1|^2, \ldots, |\kappa_1|^2, \ldots,$ (can be introduced in the initial vector by construction) are linearly independent over $\mathbb{Q}$, we can conclude that the matrix elements $|X_{11}|^2, \ldots$ uniquely determines the probability.

Undecidability

Forcing linear independence

- Theorem: If n_i are coprime integers, then $\sqrt{n_i}$ are linearly independent over $\mathbb{Q}$.
- We can then choose $\lambda_1 = \sqrt[4]{n_1}, \ldots$ and a renormalizaton factor to introduce linear independence and the case is closed. QED
- Is this an elegant solution for linear independence? Depends on the judge / no
- Any better? Only using rational numbers?

Solution using only rationals?

Observation

- Given a multivariate polynomial $f \in \mathbb{N}_0[x_1, \ldots, x_6]$, the construction tools and some other tricks give a $\lambda \in \mathbb{Q}_+$ and a QFA Q so that

$$\mathbb{P}_Q(w) = \lambda f(|X_{11}|^2, |X_{12}|^2, |X_{13}|^2, |X_{55}|^2, |X_{56}|^2, |X_{57}|^2).$$

- Does there exist a multivariate polynomial $f \in \mathbb{N}_0[x_1, \ldots, x_6]$ which is injective on rational numbers?

Finding injections

Problems

- Does there exist a multivariate polynomial $f \in \mathbb{N}_0[x_1, \ldots, x_n]$ so that $f : \mathbb{Q}^n \to \mathbb{Q}$ is an injection?
- Does there exist a multivariate polynomial $f \in \mathbb{N}_0[x_1, \ldots, x_n]$ so that $f : \mathbb{Q}^n_{\geq 0} \to \mathbb{Q}_{\geq 0}$ is an injection?
- Does there exist a bivariate polynomial $f_2 \in \mathbb{N}_0[x, y]$ so that $f_2 : \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ is an injection?
  - If we have an injection f_2 for $n = 2$ then it can be extended:

$$f_3(x, y, z) = f_2(x, f_2(y, z)), \quad f_4(x, y, z, w) = f_2(x, f_2(y, f_2(z, w))), \quad \text{etc.}$$

- Does there exist a bivariate polynomial $f \in \mathbb{N}_0[x, y]$ so that $f : \Lambda \times \Lambda \to \Lambda$ is an injection? Here $\Lambda = \{\frac{a}{5^k} \mid k \in \mathbb{N}, a \in \mathbb{N}_0, 0 \leq a \leq 5^k\}$.

Finding injections

Theorem (Cantor pairing)

$$C : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0, C(x, y) = \frac{1}{2}(x + y + 1)(x + y) + x$$

is a bijection. $C(0, 0) = 0$, $C(0, 1) = 1$, $C(1, 0) = 2$, $C(0, 2) = 3$, $C(1, 1) = 4$, $\ldots$.

Remark

- No degree 2 polynomial bijections exist other than $C(x, y)$ and $C(y, x)$ (Fueter & Pólya, 1923; Vsemirnov, 2001)
- No degree > 2 polynomial bijection $\mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ exists (P.W. Adriaans 2018)

Finding injections

Observation

$$C\left(\frac{2}{25}, \frac{11}{25}\right) = \frac{297}{625} = C\left(\frac{3}{25}, \frac{9}{25}\right).$$

More genererally, if $2a + b = 2c + d$ and $e = a + b + c + d$, then

$$C\left(\frac{a}{e}, \frac{b}{e}\right) = C\left(\frac{c}{e}, \frac{d}{e}\right).$$

Finding injections

G. Cornelissen 1999:

- Question (Harvey Friedman): Does there exist a polynomial injection $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$?
- Reply (Don Zagier): Sure, almost all complex enough polynomials will do, for example $x^7 + 3y^7$ is most likely a desired injection.

Finding injections

Theorem (Poonen 2010)

Assume that there is a homogenous polynomial $F(x, y)$ over rationals so that the rational points in the projective surface X defined as $F(x, y) = F(z, w)$ are not Zariski dense in X. Then there exists a polynomial injection $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$.

Conjecture (Bombieri-Lang)

If X is a smooth projective irreducible algebraic surface over rationals of general type. Then the set of rational points of X is not Zariski dense in X.

Remark

"General type" in the above definition refers to the Kodaira dimension. It suffices that $F(x, y)$ is separable, homogenous, and of degree at least 5 (Poonen 2010)

Remark (Cornelissen 1999)

From the (generalized) *abc*-conjecture it follows that $f(x, y) = x^n + 3y^n$ defines an injection $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ if (odd) n is large enough.

Rational Pairing Function

Theorem

Let $\Lambda = \left\{ \frac{a}{5^k} | a, k \in \mathbb{N}_0, a < 5^k \right\}$. Then $f : \Lambda \times \Lambda \to 25\Lambda$ is an injection, where:

$$f(x, y) = (x^4 + y^4)^3 + x^4$$

Note

- We can estimate the value $f(x, y)$ as

$$|(x^4 + y^4)^3 + x^4| \leq (1 + 1)^3) + 1 = 9 < 25$$

  thus $f(x, y) \in 25\Lambda$ thus an injection $f' : \Lambda \times \Lambda \to \Lambda$ can be found be introducing.a normalization factor $\frac{1}{25}$
- Injectivity follows from elementary number theory / Fermat's little theorem

QFA without radicals

Unique matrix products

As before we can use our monomorphism $\gamma : \Sigma^* \to \mathbb{Q}^{4\times 4}$ so that it is undecidable to determine if there exists a matrix in the following semigroup with two different factorizations:

$$\Gamma = \langle \{\gamma(x_j) \oplus \gamma(h(x_j)), \gamma(x_j) \oplus \gamma(g(x_j)) | 1 \leq j \leq |\Sigma|\} \rangle \subseteq \mathbb{Q}^{8\times 8}$$

Unique encoding of matrix

As before, each element of Γ is uniquely determined by six elements:

$$|X_{1,1}|, |X_{1,2}|, |X_{1,3}|, |X_{5,5}|, |X_{5,6}|, |X_{5,7}|$$

and thus by

$$\mathbf{x} = (X_{1,1}^2, X_{1,2}^2, X_{1,3}^2, X_{5,5}^2, X_{5,6}^2, X_{5,7}^2)$$

Embedding to QFA without radicals

Encoding the polynomial

As before, let $f_2(x,y) = (x^4+y^4)^3 + x^4$ and then define:

$$f_6(x_1,\dots,x_6) = f_2(x_1, f_2(x_2, f_2(x_3, f_2(x_4, f_2(x_5, x_6)))))$$

of degree $d = 12^5$

Proof Idea

Thus, $f_6(\mathbf{x}) = f_6(X_{1,1}^2, X_{1,2}^2, X_{1,3}^2, X_{5,5}^2, X_{5,6}^2, X_{5,7}^2)$ uniquely determines X

Embedding to QFA without radicals

Encoding to matrices

$$f_6(\mathbf{x}) = \sum_{i=1}^{d} T_i(\mathbf{x}) = \sum_{i=1}^{d}\sum_{j=1}^{t(i)} T_{i,j}(\mathbf{x}) = \sum_{i=1}^{d}\sum_{j=1}^{t(i)} c_{i,j} R_{i,j}(\mathbf{x}) \quad c_{i,j} \in \mathbb{N}$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{t(i)} c_{i,j} \prod_{m=1}^{i} a_{i,j,m} \quad a_{i,j,m} \in \{|X_{1,1}|, |X_{1,2}|, |X_{1,3}|, |X_{5,5}|, |X_{5,6}|, |X_{5,7}|\}$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{t(i)} \sum_{k=1}^{4} d_{i,j,k}^2 \prod_{m=1}^{i} a_{i,j,m} \quad \text{Lagrange's Theorem}$$

Embedding to Matrices

Embedding

Let us consider a particular term $c_{i,j}R_{i,j}$, of degree $i \leq \deg(f_6) = 12^5$. Note that there exists $1 \leq s, r \leq 8^i$ such that $X^{\otimes i}_{s,r} = R_{i,j}(\mathbf{x})$

Theorem

- Define $u'_{i,j,k} = d_{i,j,k} \cdot e_r \in \mathbb{Q}^{8^i}$ and $P'_{i,j} = e_s e_s^T \in \mathbb{Q}^{8^i \times 8^i}$ and then:

$$P'_{i,j} X^{\otimes i} u'_{i,j,k} = d_{i,j,k} R_{i,j}(\mathbf{x})$$

Encoding the rational pairing function

Embedding

Finally then define $P_{i,j} = \oplus_{k=1}^{4} P'_{i,j}$, $u_{i,j} = \oplus_{k=1}^{4} u'_{i,j,k}$ and $\zeta_{i,j} = \oplus_{k=1}^{4} X^{\otimes i}$

Valuation

$$\begin{aligned}
||P_{i,j}\zeta_{i,j}(X)u||^2 &= \left|\left|\bigoplus_{k=1}^{4} P'_{i,j}\zeta'_{i,j}(X)d_{i,j,k}u'\right|\right|^2 \\
&= \left(\sqrt{\sum_{k=1}^{4} d_{i,j,k}^2 R_{i,j}(\mathbf{x})^2}\right)^2 = \sum_{k=1}^{4} d_{i,j,k}^2 R_{i,j}(\mathbf{x})^2 = c_{i,j}R_{i,j}(\mathbf{x^2})
\end{aligned}$$

Final thoughts

Final embedding

With some more work we can embed the entire polynomial using tensor products and direct sums

Theorem

The injectivity problem for measure-once quantum finite automata is undecidable for $< 4 * 8^{12^5+5}$ states.

Final thoughts

Open problem

Is the knapsack variant of injectivity undecidable for MO-QFA?

Example

Given $\mathcal{Q} = (P, \{U_1, \dots U_\ell\}, \boldsymbol{x})$, does there exist distinct $k_1, \dots, k_\ell, k'_1, \dots, k'_\ell \geq 0$ such that:

$$||PU_1^{k_1} \cdots U_\ell^{k_\ell} \boldsymbol{x}||^2 = ||PU_1^{k'_1} \cdots U_\ell^{k'_\ell} \boldsymbol{x}||^2$$

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