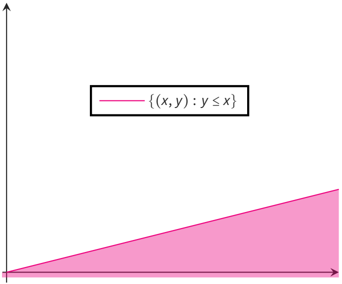
POSITIVITY PROBLEMS AND EQUALITY TESTS

George Kenison

POSITIVITY AND THE HALTING PROBLEM



Does the following loop halt?

$(x, y) = (2^{-10}, 1);$

while $y \geq x$ **do**

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end while

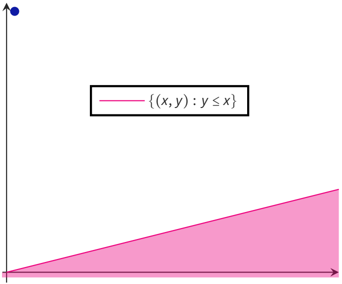
i.e., do we have

$M^n \alpha \in \{(x, y) : y \geq x\}, \forall n \in \mathbb{N}$?

The 'famous' halfplane version is commonly phrased in terms of LRSs.

Positivity

Given an integer LRS $\langle u_k \rangle_{k=0}^{\infty}$, decide whether $\forall n \in \mathbb{N}$ s.t. $u_n \geq 0$.



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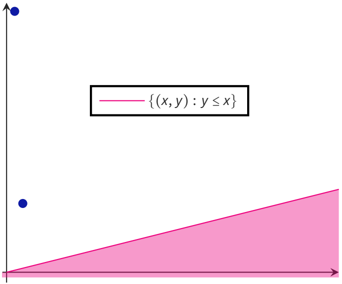

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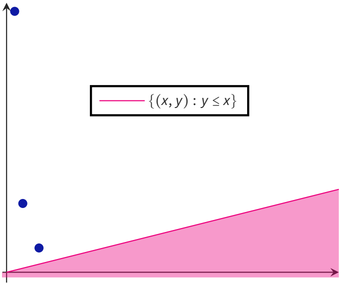

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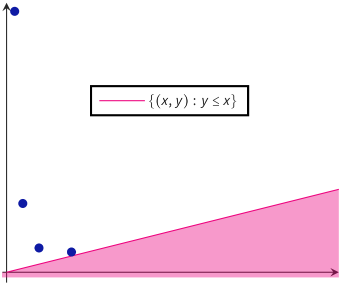

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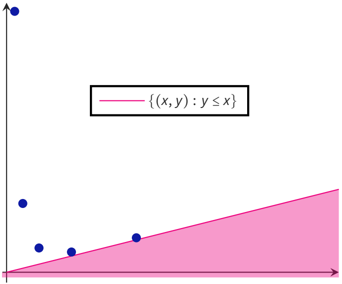

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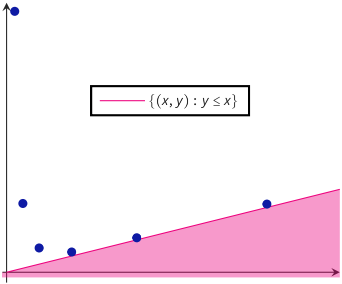

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DECISION PROBLEMS FOR RECURRENCE SEQUENCES

(Are these problems decidable?) [OW15]

Given an LRS $\langle u_n \rangle_{n=0}^{\infty}$ (over $\mathbb{Q}$ or $\mathbb{Z}$)

- Does $u_n = 0$ for some n?
- Does $u_n = 0$ for infinitely many n?
- Is $u_n \geq 0$ for all n?
- Is $u_n \geq 0$ for all but finitely many n?

An LRS satisfies a rec. relation $u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k}$ with initial values $u_0, \ldots, u_{k-1}$

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- Does $u_n = 0$ for some n? SKOLEM is open.
- Does $u_n = 0$ for infinitely many n? is decidable (Berstel–Mignotte)
- Is $u_n \geq 0$ for all n? POSITIVITY is open.
- Is $u_n \geq 0$ for all but finitely many n? ULTIMATE POSITIVITY is open.

An LRS satisfies a rec. relation $u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k}$ with initial values $u_0, \ldots, u_{k-1}$

POSITIVITY RECAP [OW14]

POSITIVITY

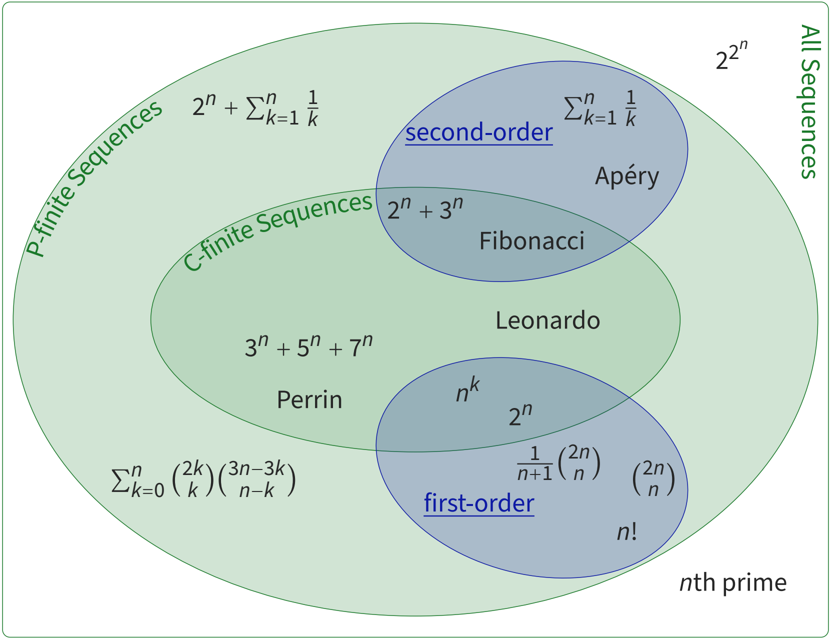
Given a recursively defined seq $\langle u_k \rangle_{k=0}^{\infty}$, decide whether $\forall n \in \mathbb{N}$ s.t. $u_n \geq 0$.

For LRSs (C-finite),

- POSITIVITY is decidable up to order 5.
- If POSITIVITY is decidable at order 6, then we can compute Lagrange constants for a large class of transcendental numbers.

How about a different family of recursively defined sequences?

THE LANDSCAPE OF P-FINITE SEQUENCES [KP11]



POSITIVITY FOR SECOND-ORDER P-FINITE SEQUENCES

[Ken+] G. Kenison, O. Klurman, E. Lefaucheux, F. Luca, P. Moree, E. C. Sertöz, J. Ouaknine, A. Whiteland, and J. Worrell. *On the Positivity Problem for second-order holonomic sequences*.

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Many more papers (e.g., [HK25; IS24; Ken24; KP10]) consider POSITIVITY and related problems in the P-finite (not necessarily second-order) setting.

SECOND-ORDER P-FINITE SEQUENCES

rational sequences $\langle u_n \rangle_{n=-1}^{\infty}$ that satisfy relations of the form

$$p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2} \quad n = 1, 2, 3, \ldots$$

where each $p_i \in \mathbb{Q}[x]$, $p_1(n), p_3(n) \neq 0$ for all $n \in \mathbb{N}$.

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rational sequences $\langle u_n \rangle_{n=-1}^{\infty}$ that satisfy relations of the form

$$u_n = b_n u_{n-1} + a_n u_{n-2} \quad \text{for } n = 1, 2, 3, \ldots$$

with $a_n := \frac{p_1(n)}{p_3(n)}$ and $b_n := \frac{p_2(n)}{p_3(n)}$.

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The solution space of the recurrence relation is a 2-dim vector space

- A soln $\langle u_n \rangle_n$ is uniquely determined by the pair u_{-1} and u_0.
- $\langle 0 \rangle_n$ is a soln.
- if $\langle u_n \rangle_n$ is a soln, then so is $\alpha \langle u_n \rangle_n = \langle \alpha u_n \rangle_n$ for every $\alpha \in \mathbb{R}$.
- if $\langle w_n \rangle_n$ and $\langle v_n \rangle_n$ are solutions, then so is $\langle w_n \rangle_n + \langle v_n \rangle_n = \langle v_n + w_n \rangle_n$.

SOLUTION SPACE AND RECURRENCE SEQUENCE PROBLEMS

Definition (Minimal solutions)

A soln $\langle u_n \rangle_n$ to $p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2}$ is *minimal* if $\langle u_n \rangle_n$ is non-trivial and there exists a second soln $\langle v_n \rangle_n$ such that $\lim_{n\to\infty} u_n/v_n = 0$.

Decision Problems for Recurrence Relations (Solution Space)

- Does a given recurrence relation admit Positive solutions?
- Does a given recurrence relation admit Minimal solutions?

Decision Problems for Recurrence Sequences

- (POSITIVITY) Is a given sequence $\langle u_n \rangle_n$ Positive? (i.e., $u_n \geq 0$, $\forall n$)
- (MINIMALITY) Is a given sequence $\langle u_n \rangle_n$ Minimal?

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- Does a given recurrence relation admit Minimal solutions? decidable

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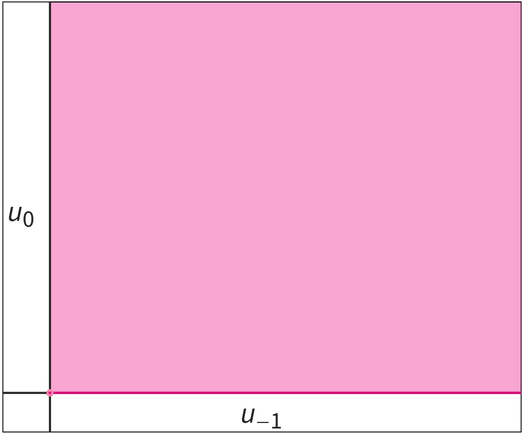
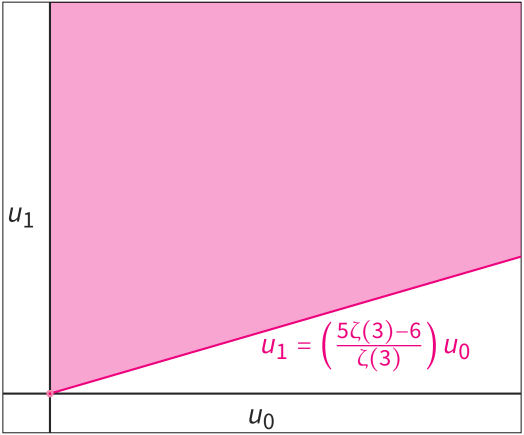
POSITIVE SOLUTIONS TO $u_{n+1} = u_n + u_{n-1}$



Set $I = \{(u_{-1}, u_0) : u_n \geq 0,\ \forall n\}$ of initial values that determine a positive solution are closed under linear combinations with positive coefficients, so I is a convex linear cone.

POSITIVE SOLUTIONS TO $n^3 u_{n+1} = (34n^3 - 51n^2 + 27n - 5)u_n - (n-1)^3 u_{n-1}$



Set $I = \{(u_0, u_1) : u_n \geq 0,\ \forall n\}$ is still a convex linear cone.

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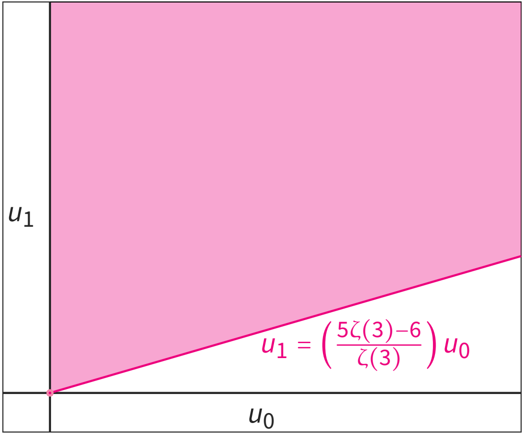

$\langle A_n \rangle_{n=0}^{\infty} = \langle 1, 5, 73, 1445, \ldots \rangle, \qquad \langle B_n \rangle_{n=0}^{\infty} = \langle 0, 6, 351/4, 62531/36, \ldots \rangle.$

Because $\frac{\zeta(3)A_n - B_n}{A_n} \to 0$ [Apé79; Poo79], $\alpha\langle \zeta(3)A_n - B_n \rangle_n$ is a *minimal* soln (substitute initial values!).

THE CONTINUED FRACTION CONNECTION

$$\mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k}{b_k} := \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots + \cfrac{b_n}{a_n}}}}$$

Theorem (Pincherle 1894 [Pin94])

- $\mathrm{K}(a_k/b_k)$ *converges in* $\hat{\mathbb{R}}$ *if and only if* $u_n = b_n u_{n-1} + a_n u_{n-2}$ *admits minimal solutions.*
- *If* $\langle w_n \rangle_{n=-1}^{\infty}$ *is a minimal solution of* $u_n = b_n u_{n-1} + a_n u_{n-2}$ *then* $\mathrm{K}(a_k/b_k)$ *converges to* $-w_0/w_{-1} \in \hat{\mathbb{R}}$.

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$$\phi = 1 + \operatorname*{K}_{k=1}^{\infty} \frac{1}{1},$$

$$\frac{4}{\pi} = 1 + \operatorname*{K}_{k=1}^{\infty} \frac{(2k-1)^2}{2},$$

$$\frac{5\zeta(3) - 6}{\zeta(3)} = \operatorname*{K}_{k=1}^{\infty} \frac{-k^6}{34k^3 + 51k^2 + 27k + 5}.$$

Theorem ([Ken+21])

Given a second-order P-finite recurrence relation $p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2}$. *We have the following:*

1. *There is a convex linear cone such that all terms of a solution* $\langle u_n \rangle_n$ *are non-negative if and only if* u_0/u_{-1} *lies in the cone.*
2. *We can decide* POSITIVITY *except when* u_0/u_{-1} *lies on the boundary of the cone.*

Further, for the class of second-order P-finite sequences, the problem of deciding POSITIVITY *Turing-reduces to that of* MINIMALITY *(i.e., a procedure to decide* MINIMALITY *implies the existence of a procedure to decide* POSITIVITY *).*

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REFERENCES [1]

[Apé79] R. Apéry. "Irrationalité de $\zeta(2)$ et $\zeta(3)$". In: *Astérisque* 61 (1979). Luminy Conference on Arithmetic, pp. 11–13.

[HK25] F. Hagihara and A. Kawamura. "The Ultimate Signs of Second-Order Holonomic Sequences". In: *52nd International Colloquium on Automata, Languages, and Programming (ICALP 2025)*. Vol. 334. 2025, 159:1–159:20.

[IS24] A. Ibrahim and B. Salvy. "Positivity Certificates for Linear Recurrences". In: *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2024, pp. 982–994. eprint: `https://epubs.siam.org/doi/pdf/10.1137/1.9781611977912.37`.

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REFERENCES [2]

[Ken+21] G. Kenison, O. Klurman, E. Lefaucheux, F. Luca, P. Moree, J. Ouaknine, M. A. Whiteland, and J. Worrell. "On Positivity and Minimality for Second-Order Holonomic Sequences". In: *International Symposium on Mathematical Foundations of Computer Science, MFCS 2021*. 2021, 67:1–67:15.

[Ken24] G. Kenison. "The Threshold Problem for Hypergeometric Sequences with Quadratic Parameters". In: *51st International Colloquium on Automata, Languages, and Programming (ICALP 2024)*. Vol. 297. 2024, 145:1–145:20.

[KP10] M. Kauers and V. Pillwein. "When can we detect that a P-finite sequence is positive?" In: *Symbolic and Algebraic Computation, International Symposium, ISSAC 2010, Munich, Germany, July 25-28, 2010, Proceedings*. 2010, pp. 195–201.

[KP11] M. Kauers and P. Paule. *The concrete tetrahedron*. Symbolic sums, recurrence equations, generating functions, asymptotic estimates. SpringerWienNewYork, Vienna, 2011, pp. x+203.

REFERENCES [3]

[OW14] J. Ouaknine and J. Worrell. "Positivity problems for low-order linear recurrence sequences". In: *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*. 2014, pp. 366–379.

[OW15] J. Ouaknine and J. Worrell. "On Linear Recurrence Sequences and Loop Termination". In: *ACM SIGLOG News* 2.2 (2015), pp. 4–13.

[Pin94] S. Pincherle. "Delle Funzioni ipergeometriche, e di varie questioni ad esse attinenti". In: *Giorn. Mat. Battaglini* 32 (1894), pp. 209–291.

[Poo79] A. van der Poorten. "A proof that Euler missed ...: An informal report". In: *The Mathematical Intelligencer* 1.4 (1979), pp. 195–203.

Sequences obeying relations of the form

$$(\alpha_1 n + \alpha_0)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}$$

are the *second-order degree-one P-finite sequences*.

Theorem ([Ken+])

Positivity and Minimality are decidable for the class of second-order degree-one P-finite sequences with two distinct characteristic roots (i.e., associated $p(x) = \alpha_1 x^2 - \beta_1 x - \gamma_1$ has two distinct rational roots).

Under these restrictive conditions, the equality checks performed as part of the decision procedure make use of the following recent results:

1. [**FR22**], for determining all polynomial relations of a given family of E-functions evaluated at a common algebraic point.
2. [**SOW**], for determining all linear relations of 1-periods.