DECISION PROBLEMS, RECURRENCE SEQUENCES, AND CONTINUED FRACTIONS

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LECTURE 1

RESEARCH PHILOSOPHY

$(x, y, z) \leftarrow (1, 2, -1).$

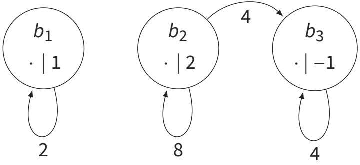
while $(*)$ **do**

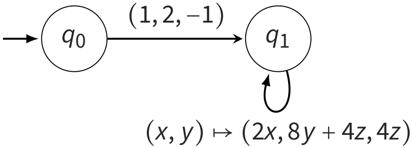
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

end while

$$x(0) = 1, y(0) = 2, z(0) = -1$$

$$\begin{cases} x(n+1) = 2x(n), \\ y(n+1) = 8y(n) + 4z(n), \\ z(n+1) = 4z(n). \end{cases}$$





WHAT IS A DECISION PROBLEM? [SIP13]

A problem where, for each valid input, the answer is either yes or no.

Decidability

A decision problem is *decidable* if there is a procedure, or algorithm, that outputs the correct answer for each valid input. (Otherwise, *undecidable*.)

Is there life on Mars? (Is this problem decidable?) [Sip13]

$$\texttt{output} := \begin{cases} \texttt{no} & \text{if life will never be found on Mars.} \\ \texttt{yes} & \text{if life will be found on Mars someday.} \end{cases}$$

Hilbert's 10th Problem (Is this problem decidable?)

Given $p \in \mathbb{Z}[x_1, x_2, \ldots, x_d]$, is there a $(y_1, y_2, \ldots, y_d) \in \mathbb{Z}^d$ s.t. $p(y_1, \ldots, y_d) = 0$?

WHAT IS A DECISION PROBLEM? [SIP13]

A problem where, for each valid input, the answer is either yes or no.

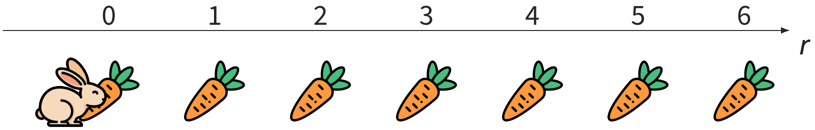
(Are these problems decidable?) (nice survey here: [OW15])

Given an LRS $\langle u_n \rangle_{n=0}^{\infty}$ (over $\mathbb{Q}$ or $\mathbb{Z}$)

- Does $u_n = 0$ for some n? SKOLEM is open.
- Does $u_n = 0$ for infinitely many n? is decidable (Berstel–Mignotte)
- Is $u_n \geq 0$ for all n? POSITIVITY is open.
- Is $u_n \geq 0$ for all but finitely many n? ULTIMATE POSITIVITY is open.

An LRS satisfies a rec. relation $u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k}$ with given initial values $u_0, \ldots, u_{k-1}$

MY FIRST LOOP SYNTHESIS PROBLEM



$r \leftarrow 0;$
while $(*)$ **do**
  $r \leftarrow r + 1;$
end while

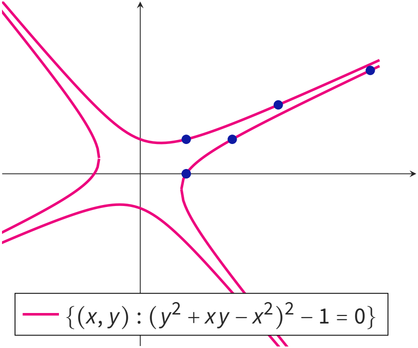
$(r, s) \leftarrow (0, 1);$
while $(*)$ **do**
  $r \leftarrow r + s;$
  $s \leftarrow s;$
end while

$(r, s) \leftarrow (0, 1);$
while $(*)$ **do**
$$\begin{pmatrix} r \\ s \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix};$$
end while

Exercise Problem (Loop Synthesis)

Given an arrangement of carrots, write a simple procedure so that a rabbit can eat a carrot at each time-step.

MOTIVATING EXAMPLE (DYNAMICAL SYSTEM VIEWPOINT)



Input: An algebraic set

$Z := \{(x, y) : (y^2 + xy - x^2)^2 - 1 = 0\}$

Find: A matrix-vector tuple* $\langle M, \boldsymbol{\alpha} \rangle$

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \;\&\; \boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

such that $\{\boldsymbol{\alpha}, M\boldsymbol{\alpha}, M^2\boldsymbol{\alpha}, \dots\} \subseteq Z$.

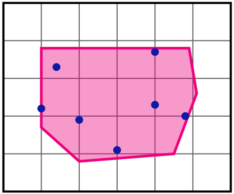
Solution:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \& \quad \boldsymbol{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

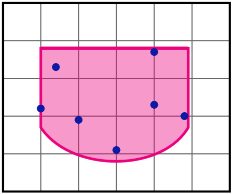
$$\boldsymbol{\alpha}, \quad M\boldsymbol{\alpha}, \quad M^2\boldsymbol{\alpha}, \quad M^3\boldsymbol{\alpha}, \quad M^4\boldsymbol{\alpha}, \dots$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \dots$$

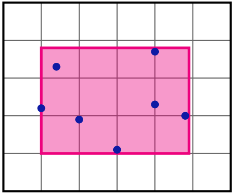
INVARIANTS AND REACHABLE STATES



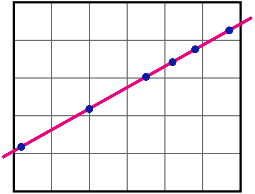
Polyhedrons



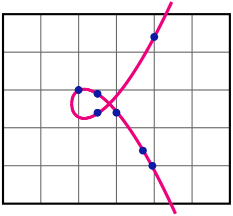
Semialgebraic sets



Intervals



Affine/linear sets



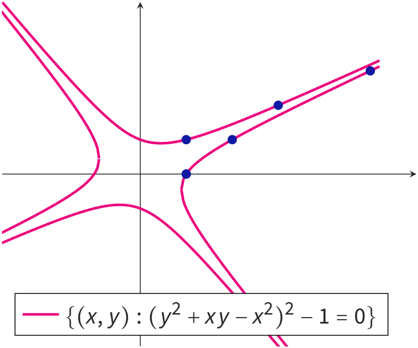
Algebraic sets

An invariant is a set that overapproximates the set of reachable states $\mathcal{O}_L$.

An algebraic set is defined by the common zeroes of a family of polynomials.

MOTIVATING EXAMPLE (LOOP SYNTHESIS VIEWPOINT)

'Strong' Loop Synthesis in [Ait+25] POPL'25.



Input: An algebraic set

$Z := \{(x, y) : (y^2 + xy - x^2)^2 - 1 = 0\}$

Find: A deterministic linear loop

$\boldsymbol{x} \leftarrow \alpha;$

while $(*)$ **do**

$\boldsymbol{x} \leftarrow M\boldsymbol{x};$

end while

with $\{\boldsymbol{\alpha}, M\boldsymbol{\alpha}, M^2\boldsymbol{\alpha}, \ldots\} \subseteq Z$,

and $\overline{\{\boldsymbol{\alpha}, M\boldsymbol{\alpha}, M^2\boldsymbol{\alpha}, \ldots\}} = Z$.

Solution:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \& \quad \boldsymbol{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Z is the *strongest* or *tightest* alg. invariant,

or the orbit closure.

This is a *best* solution wrt the *Zariski topology*.

LOOP SYNTHESIS & MARKOFF NUMBERS

Synthesise a loop with $x^2 + y^2 + z^2 - 3xyz = 0$ as an invariant.

Input:

$(x, y, z) \leftarrow (1, 1, 2);$

while $(*)$ **do**

if $(*)$ **then**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} -x \\ 3xy - z \\ -y \end{pmatrix};$$

else

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} -y \\ 3yz - x \\ -z \end{pmatrix};$$

end if

end while

([BMP25] arXiv:2509.25114)

Input: $(x, y, z) \leftarrow (1, 1, 2);$

while $(*)$ **do**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

end while

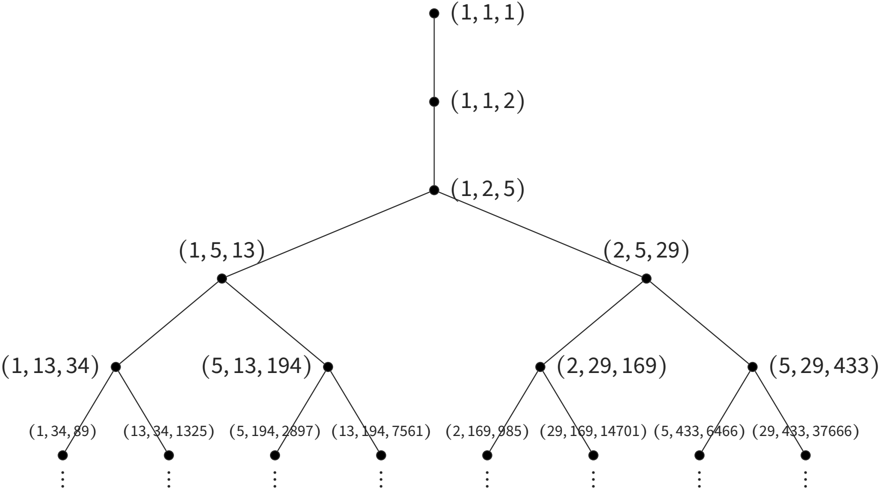
$$\alpha, \quad M\alpha, \quad M^2\alpha, \quad M^3\alpha \quad M^4\alpha, \dots \quad M^n\alpha.$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 13 \end{pmatrix}, \begin{pmatrix} 1 \\ 13 \\ 34 \end{pmatrix}, \begin{pmatrix} 1 \\ 34 \\ 89 \end{pmatrix}, \dots \begin{pmatrix} 1 \\ F_{2n-1} \\ F_{2n+1} \end{pmatrix}.$$

$$F_{2n+1} - 3F_{2n-1} + F_{2n-3} = 0$$

$$1 + (F_{2n+1})^2 + (F_{2n-1})^2 - 3F_{2n+1}F_{2n-1} = 0$$

ORDERED MARKOFF TRIPLES



$\{x, y, z \in \mathbb{Z}^3_{>0} : x^2 + y^2 + z^2 - 3xyz = 0$ and $x \leq y \leq z\}$. cf. Cassels. *An introduction to Diophantine approximation*, 1957

LOOP SYNTHESIS & MARKOFF NUMBERS

Synthesise a loop with $x^2 + y^2 + z^2 - 3xyz = 0$ as an invariant.

Input:

$(x, y, z) \leftarrow (1, 1, 2);$

while $(*)$ **do**

if $(*)$ **then**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} -x \\ 3xy - z \\ -y \end{pmatrix};$$

else

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} -y \\ 3yz - x \\ -z \end{pmatrix};$$

end if

end while

([BMP25] arXiv:2509.25114)

Input: $(x, y, z) \leftarrow (1, 1, 2);$

while $(*)$ **do**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

end while

- Are there ‘better’ (or ‘best’) loops?
  (for examples such as $x^2 + y^2 + z^2 - 3xyz = 0$)
- What about general poly invariants?
  (e.g., when does a linear loop exist?)

HOW HARD IS LOOP SYNTHESIS?

Lemma (Ait El Manssour et al. POPL'25)

Loop Synthesis over $\{\mathbb{Z}, \mathbb{Q}\}$ *is as hard as Hilbert's 10th problem over* $\{\mathbb{Z}, \mathbb{Q}\}$.

H10 asks to solve a system of polynomial equations over $R \in \{\mathbb{Z}, \mathbb{Q}\}$.

H10 is undecidable over $\mathbb{Z}$, whilst decidability is open over $\mathbb{Q}$.

Approaches to Loop Synthesis that circumvent the hardness of H10

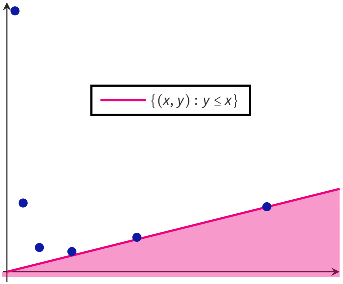
- Solve specific cases e.g., single equations e.g., Markoff, Quadratic [Hit+24]), PDBs [KKV23].
- Restrict to a finite search area and focus on complexity [Ait+25].
- Work over $\overline{\mathbb{Q}}$ rather than $\{\mathbb{Z}, \mathbb{Q}\}$ [Ait+26].

Proof strategy encodes an H10-instance $\{p_1, \ldots, p_k\}$ as the problem of finding a loop with algebraic invariant $V(p_1, \ldots, p_k, p)$

TERMINATION AND HALTING

Decision problems in program analysis and verification.

$(x, y) = (2^{-10}, 1)$;
while $y \geq x$ **do**
$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$;
end while



TERMINATION: Does L *halt* on all valid inputs?

HALTING: Does L *halt* on a given input?

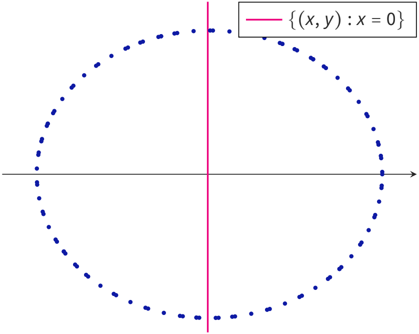
← does this loop halt?

For each $(x, y) \in \mathcal{O}_L$,

$$x^2 y - x^3 = \frac{1023}{1073741824}. \qquad (1)$$

- Properties that hold for all $(x, y) \in \mathcal{O}_L$ are (loop) invariants.
- (1) $\implies$ the loop guard is `true`.

SKOLEM AND THE HALTING PROBLEM



Does the following loop halt?

$(x, y) \leftarrow (1, 0)$.

while $x < 0 \vee x > 0$ **do**

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

end while

Alternatively, is there an $n \in \mathbb{N}$ s.t. $M^n \alpha \in \{(x, y) : x = 0\}$?

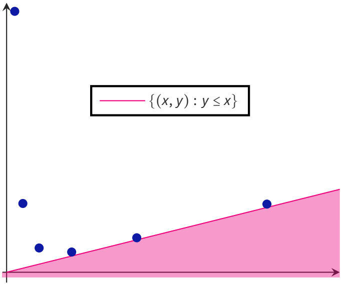
This hyperplane hitting problem is commonly phrased in terms of LRSs.

SKOLEM

Given an integer LRS $\langle u_k \rangle_{k=0}^{\infty}$, decide whether $\exists n \in \mathbb{N}$ s.t. $u_n = 0$.

inputs: characteristic polynomial and vector of initial values

POSITIVITY AND THE HALTING PROBLEM



Does the following loop halt?

$(x, y) = (2^{-10}, 1)$;

while $y \geq x$ **do**

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

end while

i.e., do we have

$M^n \alpha \in \{(x, y) : y \geq x\}, \forall n \in \mathbb{N}$?

The 'famous' halfplane version is commonly phrased in terms of LRSs.

POSITIVITY

Given an integer LRS $\langle u_k \rangle_{k=0}^{\infty}$, decide whether $\forall n \in \mathbb{N}$ s.t. $u_n \geq 0$.

inputs: characteristic polynomial and vector of initial values

BLACKBOARD INTERVAL

LECTURE 2

BLACKBOARD INTERVAL

FROM CONSTANT UPDATES TO POLYNOMIAL UPDATES

$(x, y, n) \leftarrow (5, 1, 1);$

while $(*)$ **do**

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} \frac{34n^3 - 51n^2 + 27n - 5}{n^3} & \frac{-(n-1)^3}{n^3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

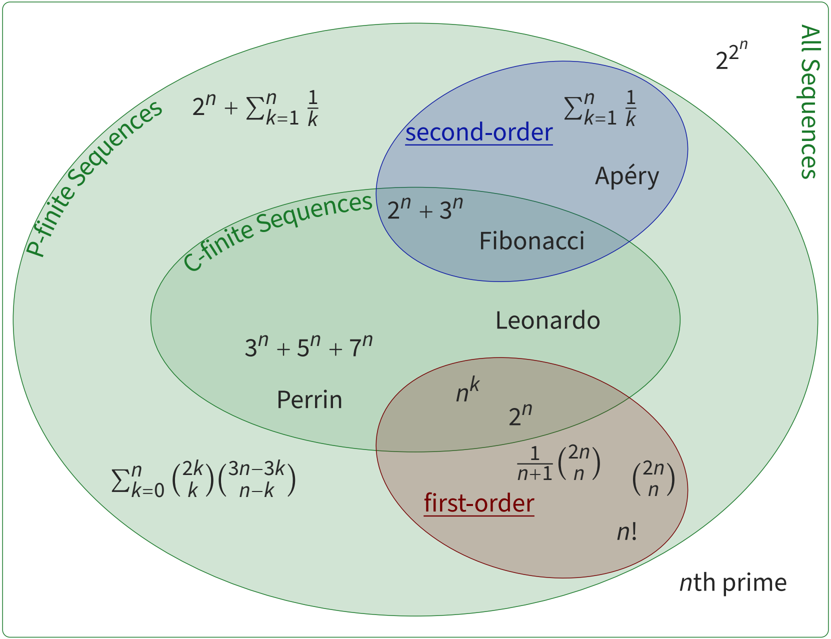
$n \leftarrow n + 1;$

end while

Loop computes Apéry's sequence $\langle 1, 5, 73, 1445, 33001, 819005, \ldots \rangle$ (OEIS A005259), which satisfies

$$n^3 x_n = (34n^3 - 51n^2 + 27n - 5)x_{n-1} - (n-1)^3 x_{n-2}.$$

THE LANDSCAPE OF P-FINITE SEQUENCES [KP11]



POSITIVITY FOR SECOND-ORDER P-FINITE SEQUENCES

[Ken+] G. Kenison, O. Klurman, E. Lefaucheux, F. Luca, P. Moree, E. C. Sertöz, J. Ouaknine, A. Whiteland, and J. Worrell. *On the Positivity Problem for second-order holonomic sequences*.

[Ken+21] G. Kenison, O. Klurman, E. Lefaucheux, F. Luca, P. Moree, J. Ouaknine, M. A. Whiteland, and J. Worrell. "On Positivity and Minimality for Second-Order Holonomic Sequences". In: *International Symposium on Mathematical Foundations of Computer Science, MFCS 2021*. 2021, 67:1–67:15.

Many more papers (e.g., [HK25; IS24; Ken24; KP10]) consider Positivity and related problems in the P-finite (not necessarily second-order) setting.

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[Ait+26] R. Ait El Manssour, G. Kenison, M. Shirmohammadi, A. Varonka, and J. Worrell. “Determination Problems for Orbit Closures and Matrix Groups”. In: *Proc. ACM Program. Lang.* 10.POPL (2026).

[BMP25] E. Bayarmagnai, F. Mohammadi, and R. Prébet. *From Affine to Polynomial: Synthesizing Loops with Branches via Algebraic Geometry*. 2025. arXiv: `2509.25114 [cs.PL]`.

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[IS24] A. Ibrahim and B. Salvy. "Positivity Certificates for Linear Recurrences". In: *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2024, pp. 982–994. eprint: `https://epubs.siam.org/doi/pdf/10.1137/1.9781611977912.37`.

[Ken24] G. Kenison. "The Threshold Problem for Hypergeometric Sequences with Quadratic Parameters". In: *51st International Colloquium on Automata, Languages, and Programming (ICALP 2024)*. Vol. 297. 2024, 145:1–145:20.

[KKV23] G. Kenison, L. Kovács, and A. Varonka. "From Polynomial Invariants to Linear Loops". In: *Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation, ISSAC 2023, Tromsø, Norway, July 24-27, 2023*. 2023, pp. 398–406.

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[KP10] M. Kauers and V. Pillwein. “When can we detect that a P-finite sequence is positive?” In: *Symbolic and Algebraic Computation, International Symposium, ISSAC 2010, Munich, Germany, July 25-28, 2010, Proceedings*. 2010, pp. 195–201.

[KP11] M. Kauers and P. Paule. *The concrete tetrahedron*. Symbolic sums, recurrence equations, generating functions, asymptotic estimates. SpringerWienNewYork, Vienna, 2011, pp. x+203.

[OW14] J. Ouaknine and J. Worrell. “Positivity problems for low-order linear recurrence sequences”. In: *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*. 2014, pp. 366–379.

[OW15] J. Ouaknine and J. Worrell. “On Linear Recurrence Sequences and Loop Termination”. In: *ACM SIGLOG News* 2.2 (2015), pp. 4–13.

[Sip13] M. Sipser. *Introduction to the Theory of Computation*. Third. 2013.

POSITIVITY RECAP

POSITIVITY

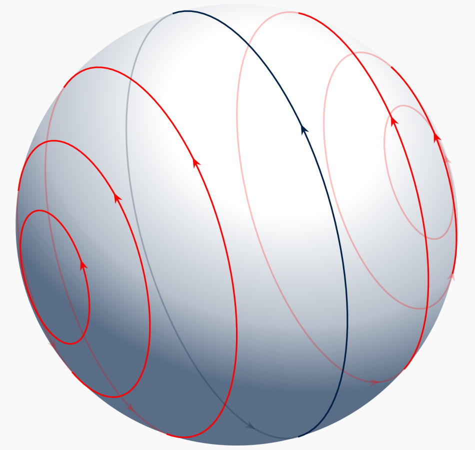
Given a recursively defined seq $\langle u_k \rangle_{k=0}^{\infty}$, decide whether $\forall n \in \mathbb{N}$ s.t. $u_n \geq 0$.

For LRSs (C-finite),

- POSITIVITY is decidable up to order 5.
- If POSITIVITY is decidable at order 6, then we can compute Lagrange constants for a large class of transcendental numbers [OW14].

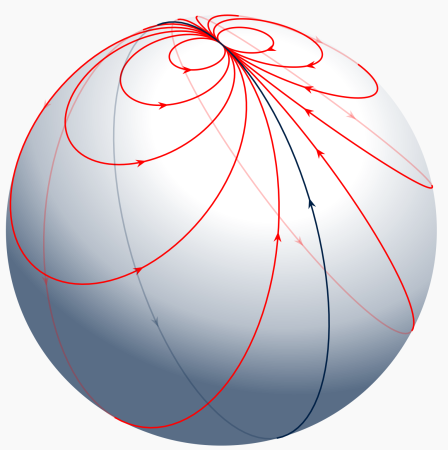
How about a different family of recursively defined sequences?

INVARIANTS ON THE RIEMANN SPHERE (ELLIPTIC TYPE)



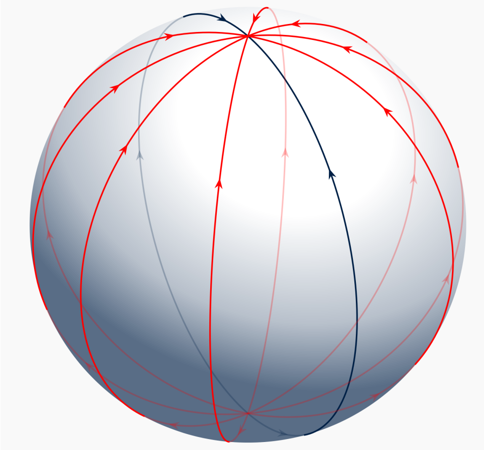
Invariants for $s(z) = \kappa/(1 + z)$ with $\kappa < -1/4$. Note, also invariants of the orbit of $\langle w_n/w_{n-1}\rangle_n$ with $w_n = w_{n-1} + \kappa w_{n-2}$.

INVARIANTS ON THE RIEMANN SPHERE (PARABOLIC TYPE)



Invariant for $s(z) = \kappa/(1 + z)$ with $\kappa = -1/4$. Note also invariants of the orbit of $\langle w_n/w_{n-1}\rangle_n$ with $w_n = w_{n-1} + \kappa w_{n-2}$.

INVARIANTS ON THE RIEMANN SPHERE (HYPERBOLIC TYPE)



Invariant for $s(z) = \kappa/(1+z)$ with $\kappa > -1/4$. Note also invariants of the orbit of $\langle w_n/w_{n-1}\rangle_n$ with $w_n = w_{n-1} + \kappa w_{n-2}$.