# Decision Problems, Recurrence Sequences, and Continued Fractions

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**Abstract**

These notes contain material from the 'chalkboard' segments of a mini-course I gave at the Institute of Mathematics of the Polish Academy of Sciences as part of the Simons Semester: "Continued fractions, fractals, ergodic theory and dynamics."

## 1 Lecture 1: Positivity for C-finite Sequences

The material in this lecture follows closely (arguably *identically*) the presentation in [OW15].

### 1.1 Preliminaries for C-finite sequences

A real-valued sequence $\langle u_n \rangle_{n=0}^{\infty}$ is said *C-finite* if it satisfies a recurrence relation of the form

$$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k}, \quad (a_k \neq 0)$$

If the preceding relation is the shortest such relation satisfied by $\langle u_n \rangle_n$, then we say that $\langle u_n \rangle_n$ has *order* $k$. The associated *characteristic polynomial* is given by $p(x) = x^k - a_1 x^{k-1} - \cdots - a_{k-1}x - a_k$. The roots of $p$ are called the *characteristic roots* of $\langle u_n \rangle_n$ and its dominant roots are those of maximal modulus.

We consider real-valued C-finite sequences and so the associated characteristic polynomials have real coefficients and non-real roots always appear in complex conjugate pairs. Thus the set of characteristic roots is of the form $\{\rho_1, \ldots, \rho_\ell, \lambda_1, \overline{\lambda_1}, \ldots, \lambda_m, \overline{\lambda_m}\}$ with each $\rho_i \in \mathbb{R}$ and each $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$.

Each C-finite sequence $\langle u_n \rangle_n$ admits an exponential-polynomial form. That is to say, there exist univariate polynomials $A_1, \ldots, A_\ell$ and $C_1, \ldots, C_m$ such that for all $n \in \mathbb{N}_0$,

$$u_n = \sum_{i=1}^{\ell} A_i(n)\rho_i^n + \sum_{j=1}^{m} \big(C_j(n)\lambda_j + \overline{C_j}(n)\overline{\lambda_j}\big).$$

For integer-valued C-finite sequences, the polynomial coefficients $A_i$ have real algebraic coefficients, whilst the polynomials $C_j$ have complex algebraic coefficients. Moreover, the degrees of each of the polynomials is at most $d-1$, where $d$ is the multiplicity of the corresponding characteristic root.

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## 1.2 The Positivity Problem

Positivity asks, given a recursively defined real-valued sequence $\langle u_n \rangle_n$, is $u_n \geq 0$ for all $n$?

In what is quite clearly a crime against terminology, a sequence is *positive* if its terms are all non-negative.

The structure of the roots of the characteristic polynomial is of paramount importance. The following classical result implies that one need only consider C-finite sequences with real positive characteristic roots in order to decide Positivity in this setting.

**Theorem 1** (Pringsheim). *Let $\langle u_n \rangle_{n=0}^{\infty}$ be a C-finite sequence with no real positive characteristic roots. Then there are infinitely many $n \in \mathbb{N}_0$ for which $u_n < 0$ and, similarly, there are infinitely many $n \in \mathbb{N}_0$ for which $u_n > 0$.*

## 1.3 A worked example

This subsection focuses on exposition and the material herein, specifically the worked example, is taken from [OW15].

**Example 2.** This example is taken wholesale from [OW15].

Let $\langle u_n \rangle_n$ be the C-finite sequence defined as follows. The $n$th term is given by

$$u_n = \frac{33}{8} + \lambda_1^n + \overline{\lambda_1^n} + 2\lambda_2^n + 2\overline{\lambda_2^n} \qquad \text{where } \lambda_1 \coloneqq \frac{-3+4\mathrm{i}}{5} \text{ and } \lambda_2 \coloneqq \frac{-7+24\mathrm{i}}{25}.$$

Our goal in this worked example is to determine whether $\langle u_n \rangle_n$ is positive. Observe that $\langle u_n \rangle_n$ is a sequence of order 5, all of its roots lie on the the unit circle $\mathbb{T} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$, and neither $\lambda_1$ nor $\lambda_2$ is a root of unity. It follows that $\{\lambda_1^n : n \in \mathbb{N}\}$ and $\{\lambda_2^n : n \in \mathbb{N}\}$ are both dense on $\mathbb{T}$.

A sufficient condition for $\langle u_n \rangle_n$ to fail the positivity test is as follows, is it possible that there is an $n \in \mathbb{N}$ such that both $\mathrm{Re}(\lambda_1^n)$ and $\mathrm{Re}(\lambda_2^n)$ are at most $-3/4$?[1] (Such a condition certainly holds if the orbit $\{(\lambda_1^n, \lambda_2^n) : n \in \mathbb{N}\}$ is a dense subset of $\mathbb{T}^2$).

However, such a condition cannot hold. We reason as follows. First, characteristic roots obey the multiplicative relation $\lambda_1^n \lambda_2^n = 1$. Second, $\{(\lambda_1^n, \lambda_2^n) : n \in \mathbb{N}\} \subset \{(z_1, z_2) \in \mathbb{T}^2 : z_1^2 z_2 = 1\} \eqqcolon \mathbf{T}$ and further, $\{(n_1, n_2) \in \mathbb{Z}^2 : \lambda_1^n \lambda_2^n = 1\}$ is a rank-1 subgroup of $\mathbb{Z}^2$ with generator $(2, 1)$. By Kronecker's Theorem [Cas57, pg.53], $\{(\lambda_1^n, \lambda_2^n) : n \in \mathbb{N}\}$ is a dense subset of $\mathbf{T}$.

We now associate to the discrete sequence $\langle u_n \rangle_n$ a continuous function $f : \mathbb{T}^2 \to \mathbb{R}$ given by $f(z_1, z_2) = z_1 + \overline{z_1} + 2z_2 + 2\overline{z_2}$. Then $u_n = f(\lambda_1^n, \lambda_2^n)$ and, by calculus, $f$ is non-negative on $\mathbf{T}$. Thus $u_n \geq 0$ for all $n \in \mathbb{N}_0$, and so we determine that $\langle u_n \rangle_n$ is positive, as desired.

Let us extend the current example to consider an order-6 C-finite sequence $\langle v_n \rangle_n$ with terms given by $v_n \coloneqq u_n - 1/2^n$. We previously established the lower bound $u_n \geq 0$. To determine whether $\langle v_n \rangle_n$ is positive, it suffices to establish the tighter lower bound $u_n \geq 1/2^n$ for all $n \in \mathbb{N}$. This will require more machinery than before since $\liminf_{n\to\infty} u_n = 0$. This claim is easily seen to hold by analysing the zeroes of the function $f$ (as above). There are two points $(z_1^*, z_2^*)$ and $(\overline{z_1^*}, \overline{z_2^*})$ in $\mathbf{T}$ where $f$ vanishes. Since $\{(\lambda_1^n, \lambda_2^n) : n \in \mathbb{N}\}$ is a dense subset of $\mathbf{T}$, we have that

$$\liminf_{n\to\infty} u_n = \liminf_{n\to\infty} f(\lambda_1^n, \lambda_2^n) = 0,$$

as desired.

We shall make use of the following simplified version of Baker's theorem on linear forms in logarithms of algebraic numbers. Our application is to bound from below $f(\lambda_1^n, \lambda_2^n)$ as a function of $n$ via lower bounds on the distance between $(\lambda_1^n, \lambda_2^n)$ and the two zeroes of $f$ in $\mathbf{T}$.

[1] If yes, then $u_n < 33/8 - 3/4 - 3/4 - 6/4 - 6/4 = -3/8 < 0$ and so $\langle u_n \rangle_n$ is not positive.

**Theorem 3** ((Simple) Baker for linear forms in 3 logarithms). *Let $\alpha_1, \alpha_2, \alpha_3$ be algebraic numbers and $b_1, b_2, b_3$ integers of absolute value at most $H$. Then for the principal branch of the complex logarithm,*

$$\Lambda \coloneqq b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3$$

*is either 0 or satisfies $|\Lambda| > H^{-C}$ where $C$ is effectively computable and depends only on $\alpha_1, \alpha_2, \alpha_3$.*

We now turn our attention to bounding the distance between $\lambda_1^n$ and $z_1^*$ as follows. Recall that $\log \alpha = \mathrm{i} \arg \alpha$ for any $\alpha \in \mathbb{T}$. Let $n$ be a natural number such that $\lambda_1^n \neq z_1^*$ and choose $m \in \mathbb{Z}$ so that $-\pi < n \arg \lambda_1 - 2\pi m \leq \pi$. Then

$$|\lambda_1^n - z_1^*| \geq \frac{1}{2} |n \arg \lambda_1 - 2\pi m - \arg z_1^*| \geq \frac{1}{\mathrm{poly}(n)} \tag{1}$$

where $\mathrm{poly}(n)$ is effectively computable and depends on $\lambda_1$ and $z_1^*$. In the above, the first inequality follows from estimating the distance of the shorter circular arc connecting $\lambda_1$ and $z_1^*$, whilst the second inequality is an application of Theorem 3.

We can now compute an $N \in \mathbb{N}$ such that $u_n \geq 1/2^n$ for all $n > N$. First, let $N_1 \in \mathbb{N}$ be such that for all $n > N_1$, we have that $(\lambda_1^n, \lambda_2^)$ is neither of the points $(z_1^*, z_2^*)$ and $(\overline{z_1^*}, \overline{z_2^*})$ where $f$ vanishes. Second, let $N_2 \in \mathbb{N}$ be such that for all $n > N_2$ we can employ the lower bound $\mathrm{poly}(n)$. Third, let $N_3 \in \mathbb{N}$ be such that for all $n > N_3$ we have that $1/\mathrm{poly}(n) \geq 1/2^n$. Finally, take $N = \max N_1, N_2, N_3 + 1$. Then $u_n \geq 1/2^n$ for all $n > N$, as desired. Thus deciding the positivity of $\langle v_n \rangle_n$ reduces to an exhaustive search where we want to determine whether $v_n \geq 0$ for each $n \in \{0, 1, \ldots, N\}$.

We conclude this section by noting that the approach to Positivityin Example 2 does not generalise to higher-order C-finite sequences. Inded, Positivityis only known to be decidable up to order 5. In fact deciding Positivityat order 6 would allow one to compute the homogeneous Diophantine approximation type of a large class of transcendental numbers (cf. [OW15]). Positivityat higher orders is (obviously) an open decision problem.

In the next lectures we shall abandon the C-finite setting for second-order recursively defined sequences.

## 2 Lecture 2: Minimality for Second-Order Recurrence Sequences

The material in this lecture follows closely (arguably *identically*) the presentation in [LW92].

### 2.1 Notation for continued fractions

The continued fraction algorithm is the function $\mathrm{K}$ mapping a pair $(\langle a_n \rangle_n, \langle b_n \rangle_n)$ onto the sequence $\langle f_n \rangle_n$ where

$$f_n \coloneqq \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots + \cfrac{b_n}{a_n}}}} = \frac{A_n}{B_n}$$

where $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$, the respective *canonical numerators* and *denominators* satisfy the recurrence relations

$$\begin{aligned} A_n &= b_n A_{n-1} + a_n A_{n-2}, \quad \text{and} \\ B_n &= b_n B_{n-1} + a_n B_{n-2} \end{aligned}$$

for each $n \in \mathbb{N}$ with initial conditions $A_{-1} = 1, A_0 = 0$, and $B_{-1} = 0, B_0 = 1$.

### 2.2 Structure of second-order solution spaces

Consider the general second-order recurrence relation

$$u_n = b_n u_{n-1} + a_n u_{n-2} \quad \text{for } n \in \mathbb{N} \tag{2}$$

where all the $a_n$ and $b_n$ are complex numbers $a_n \neq 0$ for all $n \in \mathbb{N}$.

A sequence $\langle v_n \rangle_{n=-1}^{\infty}$ is a *solution* of (2) if its elements satisfy the relation for each $n \in \mathbb{N}$. A solution $\langle v_n \rangle_n$ of (2) is uniquely determined by the pair of initial values $v_{-1}$ and $v_0$. The set of all solutions of (2) is the relation's *solution space*. The solution space has the following nice properties:

1. the sequence $\langle 0 \rangle_n$ is a solution (henceforth the *trivial solution*,
2. if $\langle v_n \rangle_n$ is a solution, then so is $\alpha \langle v_n \rangle_n = \langle \alpha v_n \rangle_n$ for every $\alpha \in \mathbb{C}$.
3. if $\langle w_n \rangle_n$ and $\langle v_n \rangle_n$ are solutions, then so is $\langle w_n \rangle_n + \langle v_n \rangle_n = \langle v_n + w_n \rangle_n$.

**Theorem 4** ([LW92, pg.192]). *The solution space of the second-order recurrence relation (2) is a vector space of dimension 2. The canonical numerators and denominators, $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$, of $\mathbf{K}_{n=1}^{\infty}(a_n/b_n)$ form a basis for the solution space.*

**Example 5.** The sequences with terms $\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ and $\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$ are easily checked to be linearly independent solutions of the second-order Fibonacci relation.

Let $\langle w_n \rangle_n$ and $\langle v_n \rangle_n$ be two linearly independent solutions of the same recurrence relation (2). Then there is no $n \in \{-1, 0, 1, \ldots\}$ for which $w_n$ and $v_n$ are both 0. This follows from the classical determinant property of such sequences (cf. [LW92, pg.196]). Thus $w_n/v_n$ is well-defined (in $\hat{\mathbb{C}}$) for each $n$.

### 2.3 Connecting minimal solutions to the convergence of continued fractions

**Definition 6** (Minimal solutions). A solution $\langle u_n \rangle_{n=-1}^{\infty}$ to (2) is *minimal* if $\langle u_n \rangle_n$ is non-trivial and there exists a second solution $\langle v_n \rangle_n$ to (2) such that $\lim_{n\to\infty} u_n/v_n = 0$. In such cases, $\langle v_n \rangle_n$ is said to be *dominant.*

Suppose that $\langle u_n \rangle_n$ is a minimal solution of (2) and $\langle v_n \rangle_n$ is a dominant solution of the same recurrence relation (2). We make the following observations:

1. the solution $\alpha \langle u_n \rangle_n$ with $\alpha \neq 0$ is also a minimal,
2. taken together, $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ form a basis set for the solution space of (2). A non-trivial solution $\langle w_n \rangle_n$ of (2) can be written as $\langle w_n \rangle_n = c_1 \langle u_n \rangle_n + c_2 \langle v_n \rangle_n$ and the condition $c_2 \neq 0$ is equivalent to $\langle w_n \rangle_n$ being a dominant solution.
3. For a non-trivial solution $\langle w_n \rangle_n$ as above, $w_n/v_n \to c_2 \neq 0$. The converse is also true, if $w_n/v_n \to c_2 \in \hat{\mathbb{C}} \setminus \{0\}$, then $\langle w_n \rangle_n$ and $\langle v_n \rangle_n$ are both dominant.

We are motivated by the following decision problems:

- Given a recurrence relation of the form (2), does the solution space contain minimal solutions?
- Minimality Problem: Given a second-order sequence $\langle u_n \rangle_n$ satisfying (2), is $\langle u_n \rangle_n$ minimal?

**Theorem 7.** *Suppose that $\langle w_n \rangle_n$ and $\langle v_n \rangle_n$, are two linearly independent solutions to (2) such that $\lim_{n\to\infty} w_n/v_n$ exists ($=: R \in \hat{\mathbb{C}}$). Then (2) admits minimal solutions.*

*Proof.* If $R = \infty$, then $\langle v_n \rangle_n$ is minimal. If $R \neq \infty$, then $\langle w_n - Rv_n \rangle_n$ is a non-trivial solution of (2) and

$$\lim_{n \to \infty} \frac{w_n - Rv_n}{v_n} = 0.$$

Thus $\langle w_n - Rv_n \rangle_n$ is minimal. □

We shall discuss a classical result of Pincherle [Pin94] that connects the existence of minimal solutions of a given recurrence relation to the convergence of the associated continued fraction.

**Non-Example 8.** Not all second-order recurrence relations admit minimal solutions. Take, for example, the relation $u_n = u_{n-2}$. The associated solution space is spanned by the pair of linearly independent solutions $\langle w_n \rangle_n = \langle 0, 1, 0, 1 \ldots \rangle$ and $\langle v_n \rangle_n = \langle 1, 0, 1, 0, \ldots \rangle$.

**Example 9.** Consider $u_n = u_{n-1} + u_{n-2}$, $n = 1, 2, \ldots$. All solutions satisfy the Binet formula $u_n = a\phi^n + b(-\phi^{-n})$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and the constants, $a$ and $b$ in the formula, are determined by substituting initial values. We have that

$$\begin{cases} u_{-1} & = a\phi^{-1} + b(-\phi), \\ u_0 & = a + b. \end{cases}$$

In this example (and for all second-order recurrence relations with constant coefficients that admit minimal solutions), it is fairly straightforward to determine if a solution is minimal. Observe that $\lim_{n\to\infty} u_n/\phi^n = 0$ if and only if $a = 0$. Thus a solution $\langle u_n \rangle_n$ of the Fibonacci recurrence is minimal if and only if $u_0/u_{-1} = b/(-b\phi) = -\phi^{-1}$. In summary, given the initial terms of a solution sequence $\langle u_n \rangle_n$ of the Fibonacci recurrence, to test whether $\langle u_n \rangle_n$ is minimal it is sufficient to test whether we have equality between $u_0/u_{-1}$ and $-\phi^{-1}$.

**Example 10.** Consider the Apéry recurrence relation

$$u_n = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} u_{n-1} - \frac{(n-1)^3}{n^3} u_{n-2}. \tag{3}$$

Since this recurrence relation has polynomial coefficients, rather than constant coefficients, we can no longer rely of a closed exponential-polynomial form for its solution sequences. However, we do know that the solution space is two-dimensional and, it just so happens, that for this example, we have the following solutions (cf. [Apé79; Poo79]):

$$\langle A_n \rangle_{n=0}^{\infty} = \langle 1, 5, 73, 1445, \ldots \rangle,$$
$$\langle B_n \rangle_{n=0}^{\infty} = \langle 0, 6, 351/4, 62531/36, \ldots \rangle.$$

van der Poorten's exposition [Poo79] demonstrates that $\zeta(3) \coloneqq \sum_{k=1}^{\infty} 1/k^3$ can be approximated by the sequence with $n$th term values $B_n/A_n$. Omitting many important details, the convergence of $B_n/A_n \to \zeta(3)$ is so fast as to ensure that $\zeta(3)$ is irrational.

We apply Theorem 7 to the above to find that $\alpha\langle B_n - \zeta(3)A_n \rangle_n$ with $\alpha \neq 0$ is a minimal solution. Thus a solution $\langle u_n \rangle_n$ to Apéry's recurrence relation (3) is minimal if and only if the ratio of its initial values satisfies $u_1/u_0 = (\zeta(3)5 - 6)/\zeta(3)$.

It now apparent that the necessary equality test to determine an instance of Minimality for the Apéry recurrence is harder than the preceding example: we may have a very good grasp on the inputs $u_0$ and $u_1$, but we are at more of a loss when it comes to the constant $(\zeta(3)5 - 6)/\zeta(3)$. Indeed, it is known that $\zeta(3)$ is irrational, but it is open whether $\zeta(3)$ is algebraic or transcendental.

We now turn to the continued fraction connection. We give a brief insight into Pincherle's theorem before giving a formal statement. Recall that the approximants of $\mathrm{K}(a_n/b_n)$ satisfy

$$f_n = \frac{A_n}{B_n} = \frac{\alpha_1 w_n + \alpha_2 v_n}{\beta_1 w_n + \beta_2 v_n} = \frac{\alpha_1 w_n/v_n + \alpha_2}{\beta_1 w_n/v_n + \beta_2}$$

where $\langle w_n \rangle_n$ and $\langle v_n \rangle_n$ are solutions of the recurrence relation (2). Notice that if $w_n/v_n \to 0$ (of minimality fame), then $\langle f_n \rangle_n$ converges to the limit $\alpha_2/\beta_2$. Pincherle's theorem is discussed in several sources (e.g., [Gau67, pg. 31], [LW92, pg. 202], [MW05, pg. 4]). The textbook [LW92] includes a formal proof of the result.

**Theorem 11** (Pincherle [Pin94]). *Let $K(a_n/b_n)$ be a continued fraction with elements $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ all from $\mathbb{C}$ and $a_n \neq 0$. Then,*

1. *$K(a_n/b_n)$ converges in $\hat{\mathbb{C}}$ if and only if the associated recurrence admits minimal solutions.*

2. *If (2) has a minimal solution then $\mathrm{K}(a_n/b_n)$ converges to $-w_0/w_{-1} \in \hat{\mathbb{C}}$.*

# 3 Lecture 3: Positivity and Minimality for Second-Order P-finite Sequences

The exposition and material in this lecture relates to the work in [Ken+21; Ken+].

A rational-valued sequence $\langle u_n \rangle_{n=-1}^{\infty}$ is a *second-order P-finite sequence* if each of its terms satisfy a recurrence relation of the form

$$p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2} \quad n = 1, 2, 3, \dots \tag{4}$$

where each $p_i \in \mathbb{Q}[x]$, $p_1(n), p_3(n) \neq 0$ for all $n \in \mathbb{N}$. An excellent textbook for relevant background material on P-finite (sometimes holonomic) sequences is [KP11]. The authors of [Ken+21] gave the following result that connects the decidability of Positivity to that Minimality for second-order P-finite sequences.

**Theorem 12** ([Ken+21]). *Given a second-order P-finite recurrence relation of the form (4). We have the following:*

1. *There is an interval $P \subseteq \mathbb{R}$ such that all the terms of any non-trivial solution $\langle u_n \rangle_n$ are non-negative if and only if $u_0/u_{-1} \in P$.*

2. *We can decide* Positivity *except when $u_0/u_{-1}$ coincides with an endpoint of $P$.*

*Further, for the class of second-order P-finite sequences, the problem of deciding* Positivity *Turing-reduces to that of* Minimality *(i.e., a procedure to decide* Minimality *implies the existence of a procedure to decide* Positivity*).*

## 3.1 Introducing polynomial continued fractions

**Example 13.** The following continued fraction expansions are well-known,

$$\begin{aligned}
\phi &= 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{1}{1}, \\
\frac{4}{\pi} &= 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(2n-1)^2}{2}, \\
\zeta(3) &= \cfrac{6}{5 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{-n^6}{34n^3 + 51n^2 + 27n + 5}}.
\end{aligned}$$

Motivated by the continued fractions of the type featured in Example 13, Bowman and MC Laughlin [BM02] coined the term *polynomial continued fractions* for the class of continued fractions whose partial quotients are (ultimately) polynomials.

PCF Equality Problem: Given a convergent polynomial continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$, determine whether its limit $f$ is equal to a particular algebraic number.

**Corollary 14** ([Ken+21]). *PCF* Equality *and* Minimality *for second-order P-finite sequences are interreducible.*

*Proof.* Suppose that $\langle u_n \rangle_n$ satisfies (4). Define $\langle v_n \rangle_n$ by $v_n := u_n \prod_{j=0}^{n} p_3(j)$. Then $\langle v_n \rangle_n$ satisfies the recurrence relation

$$v_n = p_2(n)v_{n-1} + p_1(n)p_3(n-1)v_{n-2}. \tag{5}$$

Moreover, $\langle u_n \rangle_n$ is a minimal solution of (4) if and only if $\langle v_n \rangle_n$ is a minimal solution of (3.1). Consider the polynomial continued fraction $\mathrm{K}(a_n/b_n)$ associated with (3.1) has partial quotients $a_n := p_1(n)p_3(n-1)$ and $b_n := p_2(n)$. By Theorem 11, $\langle v_n \rangle_n$ is a minimal solution of  if and only if $\mathrm{K}(a_n/b_n) \to -v_0/v_{-1}$. Thus given an oracle to determine the value of a polynomial continued fraction, one can determine whether $\langle v_n \rangle$ is minimal. Since the transformation from $\langle u_n \rangle_n$ to $\langle v_n \rangle_n$ preserves minimality, we can determine whether $\langle u_n \rangle_n$ is minimal.

Conversely, given a convergent polynomial continued fraction $\mathrm{K}(a_n/b_n)$ and $\xi$ a real algebraic number, construct a sequence $\langle v_n \rangle_{n=-1}^{\infty}$ as follows. Let $\langle v_n \rangle_n$ be a solution to the associated recurrence relation $v_n = b_n v_{n-1} + a_n v_{n-2}$ with initial values $v_{-1} = 1, v_0 = \xi$. By Theorem 11, $\langle v_n \rangle_n$ is a minimal solution if and only if $\mathrm{K}_{n=1}^{\infty}(a_n/b_n) \to -v_0/v_{-1} = \xi$. Hence if you have an oracle for determining minimality, then one can test the value of a polynomial continued fraction. $\square$

We move onto convergence results for continued fractions. The following collects together results from many sources (a good reference is [LW92]).

**Theorem 15** (Generalised Worpitzky). *Let $\kappa_n$ be a rational function in $\mathbb{Q}(n)$ and consider the continued fraction $\mathrm{K}(\kappa_n/1)$. If $\kappa_n < 0$ for all sufficiently large $n \in \mathbb{N}$, then $\mathrm{K}(\kappa_n/1)$ converges to a value in $\hat{R}$ if and only if, either*

1. *the limit $\lim_{n\to\infty} \kappa_n$ exists and is strictly greater than $-1/4$, or*
2. *$\lim_{n\to\infty} \kappa_n = -1/4$ and moreover, $\kappa_n \geq -1/4 - 1/(4n)^2 - 1/(4n\log n)^2$ for all sufficiently large $n$. Since we assume that $\kappa_n$ is a rational function, the 'eventually' statement is effective in our setting.*

**Remark 16.** The coefficient $-1/16$ of the $1/n^2$ term is best possible in the sense that if

$$\kappa_n = \begin{cases} -1/4 - \varepsilon_1/n^2 + O(1/n^3) & \text{with } \varepsilon_1 > 1/16, \text{ or} \\ -1/4 - \varepsilon_2/n + O(1/n^2) & \text{with } \varepsilon_2 > 0, \end{cases}$$

then $\mathrm{K}(\kappa_n/1)$ diverges. In essence, the convergence properties of $\mathrm{K}(\kappa_n/1)$ are connected to the convergence properties of $\mathrm{K}(\kappa)$ (where $\kappa_n \to \kappa$ as $n \to \infty$. For constant partial coefficients, the convergence properties are are related to the associated linear fractional transformations $s(z) = \kappa/(1+z)$. The continued fractions $\mathrm{K}(\kappa_n/1)$ are termed *limit hyperbolic*, *limit parabolic*, and *limit elliptic* type and are so named for the limiting linear fractional transformation.

The second convergence condition says that a limit parabolic type continued fraction converges only if the speed of convergence of $\liminf_{n\to\infty} \kappa_n$ to $-1/4$ is sufficiently fast.

**Remark 17.** Given a solution $\langle u_n \rangle_n$ to (2), we can apply a transformation $\langle u_n \rangle_n \mapsto \langle w_n \rangle_n$ such that $\langle w_n \rangle_n$ satisfies the recurrence relation

$$w_n = w_{n-1} + \kappa_n w_{n-2} \qquad \text{with } \kappa_n := \frac{a_n}{b_n b_{n-1}} \text{ where } a_n := \frac{p_1(n)}{p_3(n)} \text{ and } b_n := \frac{p_2(n)}{p_3(n)}. \tag{6}$$

Under the assumption that $\kappa_n < 0$, it is straightforward to show that this transformation preserves both positivity and minimality of sequences; that is to say, $\langle u_n \rangle_n$ is a minimal solution of (2) (resp. positive solution) if and only if $\langle w_n \rangle_n$ is a minimal solution of (6) (resp. positive solution).

The condition $\kappa_n < 0$ is no great imposition given our interest in Positivity for second-order P-finite sequences. Indeed, Positivity is inherently related to the ultimate signs of the polynomial coefficients in the recurrence relation (2) and, as seen in [Ken+21], the positivity instances where $\kappa_n > 0$ for all sufficiently large $n$ are easy to decide. Hereafter we assume that $\kappa_n < 0$ for all sufficiently large $n$. To reduce cognitive burden on the reader, we typically reduce 'sufficiently large' or 'eventually' statements to the consider of problem instances where, say, $\kappa_n < 0$ holds for all $n$.

Taken together with the Śleszyński–Pringsheim Theorem (cf. [Ken+21; LW92]), the Generalised Worpitzky Theorem (Theorem 15) leads us to the following properties of $\mathrm{K}(\kappa_n/1)$ with $\kappa_n < 0$ for all sufficiently large $n$.

1. The sequence of approximants $\langle A_n/B_n \rangle_n$ of the continued fraction $\mathrm{K}(\kappa_n/1)$ strictly decreases and converges to a finite value in $\mathbb{R}$.

2. For each $n \in \mathbb{N}$, $B_n > 0$.

Under our current assumptions, we can see that determining the positivity of sequence $\langle w_n \rangle_n$ is equivalent to determining the positivity of sequence $\langle u_n \rangle_n$. For $\langle w_n \rangle_n$ to be positive, then necessarily $w_{-1}, w_0 \geq 0$. Further $w_n \geq 0$ for each $n \in \mathbb{N}$. Using the basis property of the canonical numerator and denominator,

$$w_n = w_{-1}A_n + w_0 B_n \geq 0 \text{ if and only if } \frac{A_n}{B_n} \geq -\frac{w_0}{w_{-1}}.$$

Then the Śleszyński–Pringsheim Theorem says that we have the bound

$$-\frac{w_0}{w_{-1}} \leq \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{\kappa_n}{1}\right).$$

This bound, clearly phrased in terms of the sequence $\langle w_n \rangle_n$, is not *quite enough* to establish the interval result stated in Theorem 12 (stated in terms of $\langle u_n \rangle_n$), but we have covered the major steps.

We have one statement in Theorem 12 left to comment on. The work in [Ken+21] establishes a semi-algorithm that takes as inputs solution sequences $\langle u_n \rangle_n$ of (2). The semi-algorithm terminates for sequences that are dominant and correctly outputs 'input is a positive sequence,' or 'input is not a positive sequence' as appropriate. The semi-algorithm is non-terminating when given a minimal solution as an input.

Part of the underlying analysis of the above-mentioned semi-algorithm is an analysis of *tail sequences*, which are sequences of the form $\langle w_n/w_{n-1} \rangle_n$ and satisfy a non-linear first order recurrence relation

$$-\frac{w_n}{w_{n-1}} = -1 + \kappa_n \cdot \left(-\frac{w_{n-1}}{w_{n-2}}\right)^{-1}.$$

The cases where the recurrence relations relate to limit hyperbolic and limit parabolic type continued fractions are discussed in [Ken+21].

### 3.2 A decidability result for second-order degree-1 P-finite sequences

One of the major obstacles to a decision procedure in [Ken+21] are algorithms for performing equality tests between two real numbers.

**Definition 18** ([KZ01])**.** A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of algebraic functions with algebraic coefficients over domains in $\mathbb{R}^k$ given by polynomial inequalities with algebraic coefficients

$$\int_D g(x_1, \ldots, x_k)\, dx_1 \cdots dx_k.$$

Examples of periods include the algebraic numbers and $\pi = \int_0^\infty \frac{2}{x^2+1}\,dx$.

**Conjecture 19** ([KZ01]). *It is decidable whether two periods are equal.*

We term the class of P-finite sequences that obey second-order recurrence relations whose polynomial coefficients are all linear, i.e., relations of the form

$$(\alpha_1 n + \alpha_0)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2} \tag{7}$$

the *second-order degree-one P-finite sequences.* Subject to further restrictions, we have the following decidability result.

**Theorem 20** ([Ken+]). Positivity *and* Minimality *are decidable for the class of second-order degree-one P-finite sequences with two distinct characteristic roots (i.e., the polynomial $p(x) = \alpha_1 x^2 - \beta_1 x - \gamma_1$ associated with the recurrence relation (7) has two distinct rational roots).*

Under these restrictive conditions, the equality checks performed as part of the decision procedure make use of the following recent results:

1. An algorithm, due to Fischler and Rivoal [FR22], for determining all polynomial relations of a given family of E-functions evaluated at a common algebraic point.
2. An algorithm, due to Sertöz, Ouaknine, and Worrell [SOW25], for determining all linear relations of 1-periods.

Recall that E-functions generalise the exponential function and were introduced by Siegal, whereas 1-periods are integrals of algebraic functions of a single variable.

# 4 Summary

Let us summarise the content of the mini-course and suggest a few questions going forward.

**Lecture 1:** We discussed how common decision problems in Computer Science for simple linear loops can be posed in terms of decision problems for C-finite sequences. For C-finite sequences Positivityis decidable at low-orders, but is an open decision problem for higher-order sequences.

**Lecture 2:** We abandoned the C-finite setting and instead focused on second-order recurrence sequences. In particular, we introduced the notions of minimal and dominant solutions to second-order recurrence relations. Through Pincherle's theorem (Theorem 11), we saw a connection to the convergence of continued fractions.

**Lecture 3:** We discussed Positivity, Minimality, and PCF Equality tests for second-order P-finite sequences and the related class of polynomial continued fractions. We demonstrated recent progress on these problems in the field.

We pose the following questions.

**Question 1.** Are there classes of convergent continued fractions whose limits are sufficiently 'nice' so that we can circumvent the hard problem of testing equalities in the general setting? [Aside: the work of [Ken+] approaches this type of restriction from the *recurrence sequence* side, rather than the *continued fraction* side.]

**Question 2.** Polynomial continued fractions have 'well-behaved' partial quotients (the $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ defining $\mathop{\mathrm{K}}(a_n/b_n)$). Can we extend or generalise the methods in [Ken+21; Ken+] to other classes of 'well-behaved' continued fractions.

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