On Inequality Decision Problems for Low-Order Holonomic Sequences

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***Abstract*—An infinite sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ of real numbers is *holonomic* (also known as P-*recursive* or P-*finite*) if it satisfies a linear recurrence relation with polynomial coefficients:**

$$g_{k+1}(n)u_{n+k} = g_k(n)u_{n+k-1} + \cdots + g_1(n)u_n\,,$$

where each coefficient $g_0, \ldots, g_k \in \mathbb{Q}[n]$. Here k is the *order* of the sequence; order-1 holonomic sequences are also known as *hypergeometric* sequences. The *degree* of the sequence is the highest degree of the polynomial coefficients appearing in the recurrence relation.

A holonomic sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ is said to be *positive* if each $u_n \geq 0$, and *minimal* if, given any other linearly independent sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ satisfying the same recurrence relation, the ratio u_n/v_n converges to 0. Given two hypergeometric sequences $\langle u_n \rangle_{n\in\mathbb{N}}$ and $\langle v_n \rangle_{n\in\mathbb{N}}$, the *Hypergeometric Inequality Problem* asks whether, for all n, $u_n \leq v_n$.

In this paper, we focus on various decision problems for second-order and hypergeometric sequences, and in particular on effective reductions concerning such problems. Some of these reductions also involve certain numerical quantities (known as *periods*, *exponential periods*, and *pseudoperiods*, originating from algebraic geometry and number theory), and classical decision problems regarding equalities among these (the *Exponential Period* and *Pseudoperiod Equality Problems*).

We establish the following:

1) **For second-order holonomic sequences, the Positivity Problem reduces to the Minimality Problem.**
2) **For second-order, degree-1 holonomic sequences, the Positivity and Minimality Problems both reduce to the Equality Problems for exponential periods and pseudoperiods.**
3) **The Hypergeometric Inequality Problem reduces to the Pseudoperiod Equality Problem.**

1. Introduction

Holonomic sequences (also known as P-*recursive* or P-*finite* sequences) are infinite sequences of real (or complex) numbers that satisfy a linear recurrence relation with polynomial coefficients. The earliest and best-known example is the Fibonacci sequence, introduced by Leonardo of Pisa in the 12th century; more recently, Apéry famously made use of certain holonomic sequences satisfying the recurrence relation

$$(n+1)^3 u_{n+1} = (34n^3+51n^2+27n+5)u_n - n^3 u_{n-1} \quad (n \in \mathbb{N})$$

to prove that $\zeta(3) := \sum_{n=1}^{\infty} n^{-3}$ is irrational [4]. Holonomic sequences now form a vast subject in their own right, with numerous applications in mathematics and other sciences; see, for instance, the monographs [48], [12], [14] or the seminal paper [60].

Formally, a holonomic recurrence is a relation of the following form:

$$g_{k+1}(n)u_{n+k} = g_k(n)u_{n+k-1} + \cdots + g_1(n)u_n\,,$$

where $g_{k+1}, \ldots, g_1 \in \mathbb{Q}[n]$ are polynomials with rational coefficients. We define the *order* of the recurrence to be k, and its *degree* to be the maximum degree of the polynomials g_i. Assuming that $g_{k+1}(n) \neq 0$ for non-negative integer n, the above recurrence uniquely defines an infinite sequence $\langle u_n \rangle_{n=0}^{\infty}$ once the k initial values $u_0, \ldots, u_{k-1}$ are specified.[1] Such a sequence is said to be *holonomic*, and—in slight abuse of terminology—will be understood to inherit the *order* and *degree* of its defining recurrence. Degree-0 holonomic sequences—i.e., such that all polynomial coefficients appearing in the recurrence relation are constant—are also known as C-*finite* sequences, and first-order holonomic sequences are known as *hypergeometric* sequences.

The study of identities for holonomic sequences appears frequently in the literature. However, as noted by Kauers and Pillwein, *"in contrast,… almost no algorithms are available for inequalities"* [25]. For example, the *Positivity Problem* (i.e., whether every term of a given sequence is non-negative) for C-finite sequences is only known to be decidable at low orders, and there is strong evidence that the problem is mathematically intractable in general [44], [43]; see also [20], [34], [42]. For holonomic sequences that are not C-finite, virtually no decision procedures currently exist for Positivity, although several partial results and heuristics are known (see, for example [35], [25], [40], [59], [49], [50]).

Another extremely important property of holonomic sequences is *minimality*; a sequence $\langle u_n \rangle_n$ is minimal if, given any other linearly independent sequence $\langle v_n \rangle_n$ satisfying the same recurrence relation, the ratio u_n/v_n converges to 0. Minimal holonomic sequences play a crucial rôle, among

[1] In the sequel, it will in fact often be convenient to start the sequence at u_{-1} instead of u_0.

others, in numerical calculations and asymptotics, as noted for example in [18], [19], [17], [9], [2], [10]—see also the references therein. Unfortunately, there is also ample evidence that determining algorithmically whether a given holonomic sequence is minimal is a very challenging task, for which no satisfactory solution is at present known to exist. Indeed, one of our main results reduces Positivity to Minimality for second-order holonomic sequences.

Finally, we also consider the *Hypergeometric Inequality Problem* which asks, given two hypergeometric sequences $\langle u_n \rangle_{n\in\mathbb{N}}$ and $\langle v_n \rangle_{n\in\mathbb{N}}$, whether, for all n, $u_n \le v_n$.

The systematic study of inequality decision problems for holonomic sequences of arbitrary order is a vast undertaking. Accordingly, here we restrict our attention to first- and second-order sequences; we shall reference the following second-order recurrence equation in the rest of the paper:

$$g_3(n)u_n = g_2(n)u_{n-1} + g_1(n)u_{n-2}\,, \tag{1}$$

where $g_1, g_2, g_3 \in \mathbb{Q}[x]$.

Our main focus in the present paper is on Turing reductions among the above problems, and also on reductions to classical problems involving equality checking of certain numerical quantities (known as [univariate] *periods*, *exponential periods*, and *pseudoperiods*, which originate from algebraic geometry and number theory). As we develop in the sequel, these quantities also arise as rational linear combinations of values of hypergeometric functions evaluated at rational or algebraic parameters, or as Beta integrals evaluated at algebraic numbers. Various established conjectures appear in the literature concerning the decidability of equality checking among periods and related expressions, notably those of Kontsevich and Zagier [28]. Moreover, recent breakthrough advances in algebraic geometry and number theory have been announced which, if confirmed, suggest that in many of our cases unconditional decidability results could be obtained; this is discussed at greater length in Section 3-A.

We summarise our main results as follows:

1) For second-order holonomic sequences, the Positivity Problem reduces to the Minimality Problem (Theorem 2.1).
2) For second-order, degree-1 holonomic sequences, the Positivity and Minimality Problems both reduce to the Equality Problems for exponential periods and pseudoperiods (Theorem 5.1).
3) The Hypergeometric Inequality Problem reduces to the Pseudoperiod Equality Problem (Theorem 4.1).

Finally, taking a model-theoretic perspective, we point out that all of the above problems become decidable with the assistance of a classical Gabrielov-Vorobjov oracle, as discussed in Section 3.

2. Positivity reduces to Minimality

The goal of this section is to show that, for second-order holonomic sequences of arbitrary degree, the Positivity Problem reduces to the Minimality Problem; in other words, given an oracle for the Minimality Problem, one can decide the Positivity Problem:

Theorem 2.1. *For the class of recurrence relations*

$$u_n = b_n u_{n-1} + a_n u_{n-2} \tag{2}$$

whose coefficients are rational functions in $\mathbb{Q}(n)$, the Positivity Problem reduces to the Minimality Problem.

In our development, it is occasionally convenient to shift the indices of a given sequence by some computable amount, in effect working with a tail of the sequence. This does not affect decidability since the finitely many initial values of the sequence can always be handled separately.

A. continued fractions preliminaries

A *continued fraction*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

is defined by an ordered pair of sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ of complex numbers where $a_n \neq 0$ for each $n \in \mathbb{N}$. Herein we shall always assume that $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ are real-valued rational functions. A continued fraction *converges* to a value $f = \mathrm{K}(a_n/b_n)$ if its *sequence of approximants* $\langle f_n \rangle_{n=1}^\infty$ converges to f in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The sequence $\langle f_n \rangle_n$ is recursively defined by the following composition of linear fractional transformations. For $w \in \hat{\mathbb{R}}$, define

$$s_n(w) = \frac{a_n}{b_n + w} \text{ for each } n \in \{1, 2, \ldots\}.$$

We set $f_n := s_1 \circ \cdots \circ s_n(0)$ so that

$$f_n = \mathop{\mathrm{K}}_{m=1}^{n} \frac{a_m}{b_m} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{b_n}{a_n}}}.$$

We respectively call $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ the sequences of *partial numerators* and *partial denominators* (together the *partial quotients*) of the continued fraction $\mathrm{K}(a_n/b_n)$. Let $\langle A_n \rangle_{n=-1}^\infty$ and $\langle B_n \rangle_{n=-1}^\infty$ satisfy the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$ with initial values $A_{-1} = 1, A_0 = 0, B_{-1} = 0$, and $B_0 = 1$. As a pair, $\langle A_n \rangle_{n=-1}^\infty$ and $\langle B_n \rangle_{n=-1}^\infty$ form a basis for the solution space of the recurrence. We call $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$ the sequences of *canonical numerators* and *canonical denominators* of $\mathrm{K}(a_n/b_n)$ because $f_n = A_n/B_n$ for each $n \in \mathbb{N}$.

Two continued fractions are *equivalent* if they have the same sequence of approximants. The following theorem is attributed to Seidel in [37], [38].

Theorem 2.2. *The continued fractions $\mathrm{K}(a_n/b_n)$ and $\mathrm{K}(c_n/d_n)$ are equivalent if and only if there exists a sequence $\langle \mathfrak{r}_n \rangle_{n=0}^\infty$ with $\mathfrak{r}_0 = 1$ and $\mathfrak{r}_n \neq 0$ for each $n \in \mathbb{N}$ such that $c_n = \mathfrak{r}_n \mathfrak{r}_{n-1} a_n$ and $d_n = \mathfrak{r}_n b_n$ for each $n \in \mathbb{N}$.*

B. Śleszyński–Pringsheim continued fractions

We call a continued fraction $\mathrm{K}_{n=1}^{\infty}(c_n/d_n)$ with $|d_n| \geq |c_n|+1$ for each $n \in \mathbb{N}$ a *Śleszyński–Pringsheim continued fraction*. Let $\langle \overline{f}_n \rangle_n$ be the sequence of approximants associated with such a continued fraction. The following properties are well-known [38]. Writing $\mathbb{D}$ to denote the open unit disk in the complex plane, since $c_n/(d_n+\mathbb{D}) \subseteq \mathbb{D}$ for each $n \in \mathbb{N}$, we have that $\overline{f}_n \in \mathbb{D}$ for each $n \in \mathbb{N}$. Further, it can be shown that $\langle \overline{f}_n \rangle_n$ converges to a finite value $\overline{f}$ with $0 < |\overline{f}| \leq 1$. We will use the following convergence result. The result can be derived from the Śleszyński–Pringsheim Theorem (we reproduce the proof in [38, §3.2.4] below).

Theorem 2.3. *Let $\langle \overline{f}_n \rangle_n$ and $\langle D_n \rangle_n$ be the respective sequences of approximants and canonical denominators for a Śleszyński–Pringsheim continued fraction $\mathrm{K}_{n=1}^{\infty}(c_n/d_n)$ with $c_n < 0$ and $d_n \geq 1 - c_n$ for each $n \in \mathbb{N}$. Then*

$$D_{n+1} > D_n \geq \sum_{k=0}^{n} \prod_{m=1}^{k} (d_m - 1) \geq \sum_{k=0}^{n} \prod_{m=1}^{k} |c_m|,$$

$\langle \overline{f}_n \rangle_n$ is strictly decreasing, and $-1 < \overline{f}_n < \overline{f}_{n-1} < 0$.

Proof. We prove by induction that $\langle D_n \rangle_n$ is a strictly increasing sequence. First, $D_0 = D_0 - D_{-1} = 1$. Second, for our induction hypothesis, let us assume that $D_{n-1} - D_{n-2} > 0$ and $D_{n-2} \geq 0$. Then, using the recurrence relation and our additional assumptions on the coefficients, we have

$$\begin{aligned} D_n - D_{n-1} &= (d_n - 1)D_{n-1} - (-c_n)D_{n-2} \\ &\geq (d_n - 1)(D_{n-1} - D_{n-2}). \end{aligned}$$

Repeated application of this technique gives

$$\begin{aligned} D_n - D_{n-1} &\geq (D_{n-1} - D_{n-2})(d_n - 1) \geq \cdots \\ &\geq (D_0 - D_{-1}) \prod_{m=1}^{k} (d_m - 1) \geq \prod_{m=1}^{k} |c_m| > 0, \end{aligned}$$

from which the desired inequalities follow. We use the determinant formula (see Lemma 2.10) to obtain

$$\overline{f}_n - \overline{f}_{n-1} = -\frac{\prod_{k=1}^{n} -c_k}{D_n D_{n-1}} < 0.$$

Thus $\langle \overline{f}_n \rangle_n$ is a strictly decreasing sequence with $\overline{f}_1 = c_1/d_1 < 0$. The bounds follow from the aforementioned convergence properties of Śleszyński–Pringsheim continued fractions. □

C. second-order linear recurrences and continued fractions

Recall that a non-trivial solution $\langle u_n \rangle_{n=-1}^{\infty}$ of the recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$ is *minimal* provided that, for all other linearly independent solutions $\langle v_n \rangle_{n=-1}^{\infty}$ of the same recurrence, we have $\lim_{n\to\infty} u_n/v_n = 0$. Since the vector space of solutions has dimension 2, it is equivalent for a sequence $\langle u_n \rangle_{n=-1}^{\infty}$ to be minimal for there to exist a linearly independent sequence $\langle v_n \rangle_{n=-1}^{\infty}$ satisfying the above property. In such cases the solution $\langle v_n \rangle_n$ is called *dominant*.

If $\langle u_n \rangle_n$ is minimal then all solutions of the form $\langle c u_n \rangle_n$ where $c \neq 0$ are also minimal. Note that if $\langle y_n \rangle_n$ and $\langle z_n \rangle_n$ are linearly independent solutions of the above recurrence such that $y_n/z_n \sim C \in \hat{\mathbb{R}}$ then the recurrence relation has a minimal solution [37]. If $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are respectively minimal and dominant solutions of the recurrence, then together they form a basis of the solution space.

Remark 2.4. When a second-order recurrence relation admits minimal solutions, it is often beneficial (from a numerical standpoint) to provide a basis of solutions where one of the elements is a minimal solution. Such a basis is used to approximate any element of the vector space of solutions: taking $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ as above, a general solution $\langle w_n \rangle_n$ is given by $w_n = \alpha_1 u_n + \alpha_2 v_n$ and is therefore dominant unless $\alpha_2 = 0$.

Let $\langle u_n \rangle_{n=-1}^{\infty}$ be a non-trivial solution of the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. If $u_{n-1} \neq 0$ then we can rearrange the relation to obtain

$$-\frac{u_{n-1}}{u_{n-2}} = \cfrac{a_n}{b_n - \cfrac{u_n}{u_{n-1}}} = s_n\left(-\frac{u_n}{u_{n-1}}\right) \tag{3}$$

for each $n \in \mathbb{N}$. In the event that $u_{n-2} = 0$ we take the usual interpretation in $\hat{\mathbb{R}}$. Since $\langle u_n \rangle_n$ is non-trivial and $a_n \neq 0$ for each $n \in \mathbb{N}$, the sequence $\langle u_n \rangle_n$ does not vanish at two consecutive indices. Thus if $u_{n-1} = 0$ then $u_{n-2}, u_n \neq 0$ and so both the left-hand and the right-hand sides of the last equation are well-defined in $\hat{\mathbb{R}}$ and are equal to 0. Thus the sequence with terms $-u_n/u_{n-1}$ is well-defined in $\hat{\mathbb{R}}$ for each $n \in \mathbb{N}$. A sequence $\langle t_n \rangle_{n=0}^{\infty}$ with each term given by $t_n := -u_n/u_{n-1}$ (where $\langle u_n \rangle_n$ is non-trivial) is called a *tail sequence*.

The next theorem due to Pincherle [51] connects the existence of minimal solutions for a second-order recurrence to the convergence of the associated continued fraction (see also [18], [37], [8]).

Theorem 2.5 (Pincherle)**.** *Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be real-valued sequences such that each of the terms a_n is non-zero. First, the recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$ has a minimal solution if and only if the continued fraction $\mathrm{K}(a_n/b_n)$ converges. Second, if $\langle u_n \rangle_n$ is a minimal solution of this recurrence then the limit of $\mathrm{K}(a_n/b_n)$ is $-u_0/u_{-1}$. As a consequence, the sequence of canonical denominators $\langle B_n \rangle_{n=-1}^{\infty}$ is a minimal solution if and only if the value of $\mathrm{K}(a_n/b_n)$ is $\infty \in \hat{\mathbb{R}}$.*

Remark 2.6. The convergence properties of continued fractions whose partial quotients are polynomials has long fascinated researchers. It is notable that the sequence of partial denominators in the simple continued fraction expansion $\pi = 3 + \mathrm{K}_{n=1}^{\infty}(1/b_n)$ beginning $\langle b_n \rangle_n = \langle 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots \rangle$ behaves erratically. In contrast, Lord Brouncker (as reported by Wallis in [56][2]) gave a

[2]See the translation by Stedall [57].

continued fraction expansion for $4/\pi$ as follows:

$$\frac{4}{\pi} = 1 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{(2n-1)^2}{2}.$$

Apéry's constant $\zeta(3)$ also has a regular continued fraction expansion (see [54])

$$\zeta(3) = \frac{6}{5 + \mathrm{K}_{n=1}^{\infty}(-n^6/(34n^3+51n^2+27n+5))}$$

whose partial quotients are ultimately polynomials. Motivated by such constructions, Bowman and Mc Laughlin [6] (see also [39]) coined the term *polynomial continued fraction*. A polynomial continued fraction $\mathrm{K}(a_n/b_n)$ has algebraic partial quotients such that for sufficiently large $n \in \mathbb{N}$ we have a_n, b_n are determined by polynomials $\mathbb{Q}[n]$. The evaluation of polynomial continued fractions whose partial quotients have low degrees appears in [47], [37], [8]. For $\deg(a_n) \le 2$ and $\deg(b_n) \le 1$, Lorentzen and Waadeland [37, §6.4] express the polynomial continued fraction $\mathrm{K}(a_n/b_n)$ as a ratio of the value of two hypergeometric functions with algebraic parameters evaluated at an algebraic point. However, those authors do not cover all cases at low degrees; for example, the polynomial continued fraction $\mathrm{K}_{n=1}^{\infty}\frac{-n(n+1)}{2n+1}$ corresponding to the recurrence relation $(n+1)u_n = (2n+1)u_{n-1} - (n+1)u_{n-2}$ is not treated. Indeed the presentation in [37] does not handle cases where the corresponding recurrence has a single repeated characteristic root—the above is one such example with its associated characteristic polynomial $x^2 - 2x + 1 = (x-1)^2$. We extend their method in our proof of Theorem 5.7.

We call the problem of determining whether a given convergent polynomial continued fraction is equal to a particular algebraic number the *PCF Equality Problem*. The proof of the following corollary is given in Appendix A.

Corollary 2.7. *The PCF Equality Problem and the Minimality Problem for second-order holonomic sequences are interreducible.*

Determining whether a given continued fraction converges has attracted much attention (historical accounts are given in [37], [38]). We state the following useful specialisation.

Theorem 2.8. *Let $\mathrm{K}(a_n/1)$ be a continued fraction with $\langle a_n \rangle_n$ a rational function in $\mathbb{Q}(n)$.*

First, if $a_n < 0$ for sufficiently large $n \in \mathbb{N}$, then $\mathrm{K}(a_n/1)$ converges to a value in $\hat{\mathbb{R}}$ if and only if, either (i) $\lim_{n\to\infty} a_n$ exists and is strictly above $-1/4$, or (ii) $\lim_{n\to\infty} a_n = -1/4$ and moreover $a_n \ge -1/4 - 1/(4n)^2 - 1/(4n\log n)^2$ for all sufficiently large n.

Second, if $a_n > 0$ for sufficiently large $n \in \mathbb{N}$, then $\mathrm{K}(a_n/1)$ converges to a value in $\hat{\mathbb{R}}$ if and only if the associated Stern–Stolz series $\sum_{n=1}^{\infty} \left| \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}} \right|$ diverges to ∞.

Remark 2.9. Recall from Theorem 2.5 that recurrence (4) has minimal solutions if and only if the continued fraction $\mathrm{K}(\kappa_n/1)$ converges. The question of whether the continued fraction $\mathrm{K}(a_n/1)$ converges is not so straightforward when a_n converges to $-1/4$. Suppose that a_n converges to $-1/4$ from below; then $\mathrm{K}(a_n/1)$ converges only if a_n converges with sufficient speed [37], [38]. Herein (Lemma 2.12 below) we prove that this criterion is crucial in determining whether a recurrence relation has non-trivial positive solutions.

The fact that the coefficients -1/16 in Theorem 2.8 are best possible in the above is discussed in [23], [22], [37], [38]. For example, if $a_n = -1/4 - \varepsilon/n^2 + \mathscr{O}(1/n^3)$ where $\varepsilon > 1/16$, or $a_n = -1/4 - \varepsilon_1/n + \mathscr{O}(1/n^2)$ where $\varepsilon_1 > 0$ then the continued fraction $\mathrm{K}(a_n/1)$ diverges. We note that later independent work by Kooman [30], [29] establishes the same results (as a consequence of his results for linear recurrence sequences).

The following determinant lemma below is well-known (see, for example, [37, Lemma 4, §IV]).

Lemma 2.10. *Suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are both solutions to the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. Then*

$$u_n v_{n-1} - u_{n-1} v_n = (u_0 v_{-1} - u_{-1} v_0) \prod_{k=1}^{n} (-a_k).$$

D. reduction argument

Recall our motivation for holonomic recurrences of the form (1). We note that we can safely assume that none of the polynomial coefficients are identically zero (see Section 4). Moreover by considering a shifted recurrence relation we can also assume that each polynomial coefficient has constant sign,[3] and has no roots for $n \ge 0$. Additionally we can assume that $\mathrm{sign}(g_3) = 1$. Thus we define the *signature* of a recurrence relation (1) (or its normalisation (2)) as the ordered pair $(\mathrm{sign}(g_2(n)), \mathrm{sign}(g_1(n)))$.

It is useful to consider subcases determined by the signature of the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. The Positivity Problem is trivial when the signature of the recurrence is either $(+,+)$ or $(-,-)$. It remains to consider the cases $(-,+)$ and $(+,-)$.

Let $\langle u_n \rangle_n$ satisfy a recurrence with signature $(-,+)$. Then a simple substitution argument gives

$$\begin{aligned} u_{2n} = (b_{2n}b_{2n-1} + a_{2n} + a_{2n-1}b_{2n}/b_{2n-2})u_{2n-2} \\ - (a_{2n-1}a_{2n-2}b_{2n}/b_{2n-2})u_{2n-4}. \end{aligned}$$

The sequence of odd terms $\langle u_{2n-1} \rangle_n$ also satisfies a similar recurrence relation with with signature $(+,-)$. Thus the Positivity Problem for the $(-,+)$ case reduces to determining the Positivity Problem for two recurrences with signature $(+,-)$. It is worth noting here that this method does not preserve the degrees of the coefficients.

We come to the final case: recurrences with signature $(+,-)$. Let $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ be the canonical solutions as above. In this case $A_1 = a_1 < 0$ and so one can assume that $u_0 > 0$. We defer the discussion of the positivity of $\langle B_n \rangle_{n=-1}^{\infty}$

[3] We denote the sign of a number alternately as belonging to $\{1, 0, -1\}$ or (if zero is excluded) as belonging to $\{+, -\}$.

until later. It is useful to normalise recurrence (2). Set $b_0 = 1$, $\kappa_n := a_n/(b_n b_{n-1})$, and consider

$$w_n = w_{n-1} + \kappa_n w_{n-2}. \tag{4}$$

Then $\langle w_n \rangle_n$ with $w_{-1} = u_{-1}$ and $w_n := u_n/(\prod_{k=1}^n b_k)$ is a solution to (4) if and only if $\langle u_n \rangle_n$ is a solution to (2). We note minimality, positivity, and signature $(+,-)$ are invariant under this transformation. Such properties follow from our assumption that each $b_n > 0$ and the equivalence transformations for continued fractions in Theorem 2.2.

In light of this reduction, the next result is an immediate corollary of Theorem 2.5 and Theorem 2.8.

Corollary 2.11. *Given a recurrence relation of the form* (2)*, it is decidable whether the recurrence admits a minimal solution.*

In the work that follows we split the $(+,-)$ case into subcases depending on the limit $\kappa = \lim_{n\to\infty} \kappa_n$.

Lemma 2.12. *Suppose that the continued fraction* $\mathrm{K}(\kappa_n/1)$ *diverges. Then there are no non-trivial positive solutions to recurrence* (4)*.*

Proof. Suppose, for a contradiction, that $\langle w_n \rangle_n$ is a positive sequence and non-trivial solution of recurrence (4). Let $\langle t_n \rangle_n$ be the tail sequence associated with $\langle w_n \rangle_n$. The terms in $\langle t_n \rangle_n$ are well-defined in $\hat{\mathbb{R}}$ as two consecutive terms in $\langle w_n \rangle_n$ cannot both vanish. For each $n \in \mathbb{N}$, $t_n \neq 0$ since then $w_n = 0$ and $w_{n+1} = \kappa_{n+1} w_{n-1} < 0$. Similarly, there is no $n \in \mathbb{N}$ for which $t_n = \infty$ as then $w_{n-1} = 0$ and $w_n < 0$. Thus $\langle t_n \rangle_n$ is a sequence of negative values in $\mathbb{R}$.

Let $\langle \breve{w}_n \rangle_n$ be a solution sequence of recurrence (4) such that $\breve{w}_{-1} = w_{-1}$ and $\breve{w}_0 > w_0$. In addition, let $\langle \breve{t}_n \rangle_n$ be the tail sequence associated with $\langle \breve{w}_n \rangle_n$. We make the following observations. First, we make the comparison $\breve{w}_1 = \breve{w}_0 + \kappa_1 \breve{w}_{-1} > w_1 > 0$. Second, by Lemma 2.10 $\breve{t}_1 \in \mathbb{R}$ and $\breve{t}_1 > t_1$. Then proceed by induction on n: use Lemma 2.10 to prove that $\breve{t}_n > t_n$, and further, $\breve{w}_n > w_n$ for each $n \in \mathbb{N}$.

Notice that $\langle \breve{w}_n - w_n \rangle_n$ is a positive sequence and non-trivial solution of recurrence (4). For each $n \in \{-1, 0, \ldots\}$ we have $\breve{w}_n - w_n = (\breve{w}_0 - w_0) B_n$ and so conclude that $B_n > 0$ for each $n \in \mathbb{N}$.

Let $\langle f_n \rangle_n$ be the sequence of approximants associated with $\mathrm{K}(\kappa_n/1)$. From Lemma 2.10 and our conclusion that $B_n > 0$ for each $n \in \mathbb{N}$, we obtain

$$f_n - f_{n-1} = -\frac{\prod_{k=1}^n -\kappa_k}{B_n B_{n-1}} < 0.$$

Thus $\langle f_n \rangle_n$ is monotonic and therefore convergent in $\hat{\mathbb{R}}$, a contradiction to the divergence of $\mathrm{K}(\kappa_n/1)$. ☐

In what follows, *"eventually"* statements shall always assume that a property holds for each $n + N$ where $N \in \mathbb{N}$ is a fixed computable constant. Our assumption on the signature means that $a_{n+N} < 0$ and $b_{n+N} > 0$ for each $n \in \mathbb{N}$. Without loss of generality, we can take $N = 0$ in the upcoming statements and results by considering tails of continued fractions as appropriate.

From Remark 2.9 and Lemma 2.13 (below) we realise the boundary for a recurrence relation of the form (4) to admit positive solutions. The proof of Lemma 2.13 uses standard analytic tools for continued fractions of limit parabolic type with a particular choice of parameter sequence $\langle \mathfrak{g}_n \rangle_n$. More general discussions are given in [24], [37], [36].

Lemma 2.13. *Suppose that eventually*

$$\kappa_n \geq -1/4 - 1/(4n)^2 - 1/(4n \log n)^2. \tag{5}$$

Then the sequence of approximants of the continued fraction $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ *is strictly decreasing and converges to a finite value.*

Proof. Without loss of generality we assume that (5) holds for each $n \in \mathbb{N}$. Let $\mathfrak{g}_0 = 1$ and $\mathfrak{g}_n := 1/2 + 1/(4n) + 1/(4n \log n)$ for each $n \in \mathbb{N}$. The continued fractions $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ and

$$b_N \mathfrak{g}_0 \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{\kappa_n/(\mathfrak{g}_{n-1}\mathfrak{g}_n)}{1/\mathfrak{g}_n} \right) \tag{6}$$

are equivalent; one can prove this assertion by applying Theorem 2.2 with the transformation choice $\mathfrak{r}_n = 1/(b_n \mathfrak{g}_n)$ for each $n \in \mathbb{N}$. Then, by assumption, $|\kappa_n| \leq \mathfrak{g}_{n-1}(1 - \mathfrak{g}_n)$ for each $n \in \mathbb{N}$. Thus

$$1 - \frac{\kappa_n}{\mathfrak{g}_{n-1}\mathfrak{g}_n} = \frac{\mathfrak{g}_{n-1}\mathfrak{g}_n - \kappa_n}{\mathfrak{g}_{n-1}\mathfrak{g}_n} \leq \frac{1}{\mathfrak{g}_n}.$$

We deduce that the partial numerators and denominators in (6) satisfy the assumptions in Theorem 2.3. Thus the sequence of approximants $\langle \overline{f}_n \rangle_{n=1}^{\infty}$ associated with (6) is strictly decreasing and converges to a finite value. The desired result follows. ☐

Lemma 2.14. *Suppose that* $\langle w_n \rangle_{n=-1}^{\infty}$ *is a solution to* (4) *with signature* $(+,-)$ *such that* (5) *holds for each* $n \in \mathbb{N}$*. Let* $\langle f_n \rangle_n$ *be the sequence of approximants for the associated continued fraction* $\mathrm{K}(\kappa_n/1)$*. Assume that* $w_{-1} > 0$*. Given* $m \in \mathbb{N}$*, we have that* $-w_0/w_{-1} < f_m$ *if and only if* $w_m > 0$*.*

Proof. Let $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$ be the sequences of canonical numerators and denominators associated to $\mathrm{K}(\kappa_n/1)$. The continued fractions $\mathrm{K}(\kappa_n/1)$ and (6) are equivalent; in addition, the latter is a Śleszyński–Pringsheim continued fraction and so each of the terms in its associated sequence of canonical denominators is non-negative (by Theorem 2.3). The transformation between these two continued fractions preserves the positivity property and so we deduce that each term in $\langle B_n \rangle_n$ is also non-negative.

For each $n \in \mathbb{N}$, $w_n = w_{-1} A_n + w_0 B_n$. Since $B_n > 0$, $-w_0/w_{-1} < A_n/B_n = f_n$ if and only if $w_n > 0$, as desired. ☐

We are now in the position to characterise positive solutions.

Proposition 2.15. *Suppose that* $\langle w_n \rangle_{n=-1}^{\infty}$ *is a solution of recurrence* (4) *with signature* $(+,-)$ *such that* (5) *holds for all* $n \in \mathbb{N}$*. First, the continued fraction* $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ *converges to a finite limit* $f < 0$*. Second, the sequence* $\langle w_n \rangle_{n=-1}^{\infty}$ *with* $w_{-1}, w_0 > 0$ *is positive if and only if* $-w_0/w_{-1} \leq f$*.*

Proof. By Lemma 2.13, $\mathrm{K}(a_n/b_n)$ and (6) are equivalent continued fractions. The former converges to a negative value $f \in \mathbb{R}$ because the latter is a Śleszyński–Pringsheim continued fraction that satisfies the assumptions in Theorem 2.3.

Let $\langle w_n \rangle_{n=-1}^{\infty}$ be a solution to recurrence (4). For each $n \in \mathbb{N}$, $w_{-1}, w_0, \ldots, w_n > 0$ if and only if $-w_0/w_{-1} < f_n$ by Lemma 2.14. Thus $\langle w_n \rangle_{n=-1}^{\infty}$ is positive if and only if $-w_0/w_{-1} \leq f$. □

The difficulty one encounters when determining positivity arises when $-w_0/w_{-1}$ is equal to the value f. In other words, we can decide positivity of dominant sequences. Indeed, one can always detect if a non-trivial solution $\langle w_n \rangle_n$ is not positive, i.e., $-w_0/w_{-1} > f$ by computing a sufficient number of terms until one finds an $N \in \mathbb{N}$ such that $w_N < 0$. The dominant positive sequences are considered in the following proposition whose proof is delayed to Appendix B.

Proposition 2.16. *Let $\langle w_n \rangle_{n=-1}^{\infty}$ be a non-trivial solution of* (4) *with signature $(+,-)$ and suppose that* (5) *holds for each $n \in \mathbb{N}$. Then one can detect if $-w_0/w_{-1} < f$.*

We deduce that if one can decide whether a holonomic sequence $\langle u_n \rangle_n$ that solves recurrence (2) is minimal, then one can decide whether $\langle u_n \rangle_n$ is a positive solution.

Proof of Theorem 2.1. Assume that one has an oracle for the Minimality Problem for solutions $\langle u_n \rangle_{n=-1}^{\infty}$ to recurrences of the form (2). We can assume without loss of generality that the recurrence has signature $(+,-)$. As previously mentioned, the problem of determining the positivity of solutions $\langle u_n \rangle_n$ of (2) is equivalent to the problem of determining the positivity of solutions $\langle w_n \rangle_n$ of (4).

Consider a recurrence relation of the form (4) with signature $(+,-)$. If $\mathrm{K}(\kappa_n/1)$ diverges then, by Lemma 2.12, the recurrence has no non-trivial positive solution sequences. Thus we need only consider cases when $\mathrm{K}(\kappa_n/1)$ converges.

When κ_n converges to $-1/4$ from below sufficiently quickly or to a limit $\kappa > -1/4$ the associated continued fraction converges monotonically (Lemma 2.13). We use this observation in Proposition 2.15 to deduce that the recurrence admits positive solution sequences. It is then left to detect such solutions. One can easily determine whether a solution is trivial. If $\langle w_n \rangle_n$ is minimal then, by Theorem 2.5, $-w_0/w_{-1} = f$. Hence a minimal solution is positive by Proposition 2.15. If $\langle w_n \rangle_n$ is dominant then, by Proposition 2.16, one can detect if $-w_0/w_{-1} < f$. The case $-w_0/w_{-1} > f$ can also be detected as the sequence has a negative term. This process is equivalent to determining whether $\langle w_n \rangle_n$ is positive, from which the desired result follows. □

3. Periods and Related Expressions

We recall the definitions of *periods* and *exponential periods* due to Kontsevich and Zagier, and introduce the notion of *pseudoperiod*, which is formally closely related to them. We also discuss some classical decision problems concerning these numerical quantities, which turn out to be intimately connected to the decision problems on holonomic sequences considered in this paper.

A. periods and the Kontsevich–Zagier Conjecture

Kontsevich and Zagier's seminal paper [28] defines a *period* as a complex number that can be obtained as the value of an integral of an algebraic function over a semialgebraic domain. That is to say, the real and imaginary parts of the number can be written as absolutely convergent integrals of the form

$$\int_\sigma f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

where f is an algebraic function and the domain $\sigma \subseteq \mathbb{R}^n$ is given by polynomial inequalities with algebraic coefficients. The set of periods $\mathscr{P}$ forms a countable subring of $\mathbb{C}$ and it is easily seen that $\overline{\mathbb{Q}} \subset \mathscr{P} \subset \mathbb{C}$. Two examples are:

$$\log(\alpha) = \int_1^\alpha \frac{1}{x}\, dx \text{ for } \alpha \in \overline{\mathbb{Q}} \quad \text{and} \quad \pi = \int_0^\infty \frac{2}{x^2+1}\, dx.$$

Given two algebraic numbers α and β, the problem of determining algorithmically whether $\alpha = \beta$ is known to be decidable. The decidability of the equality of two periods—that is, a decision procedure determining whether two periods (given by two explicit integrals) are equal—is currently open. The next conjecture, due to Kontsevich and Zagier [28, Conjecture 1] would entail that equality of periods is decidable.

Conjecture 3.1. Suppose that a period has two integral representations. One can pass between the representations via a finite sequence of admissible transformations where each transformation preserves the structure that all functions and domains of integration are algebraic with coefficients in $\overline{\mathbb{Q}}$. The admissible transformations are: linearity of the integral, a change of variables, and Stokes's formula.

It is currently not known whether Euler's number e is a period. The following more general notion was introduced in [28] to extend the definition of period to a larger class containing e. An *exponential period* is a complex number that can be written as an absolutely convergent integral of the form

$$\int_\sigma \mathrm{e}^{-f(x_1,\ldots,x_n)} g(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

where f and g are algebraic functions with algebraic coefficients and the domain $\sigma \subseteq \mathbb{R}^n$ is a semialgebraic set defined by polynomials with algebraic coefficients. Conjecture 3.1 is predicted to generalise to exponential periods in [28].

An overview discussing both periods and exponential periods can be found in [55].

The conjecture of Kontsevich and Zagier is quite general and powerful. However, the specific periods and exponential periods that we encounter in this paper are single-dimensional (in the sense that the integrand is univariate) and moreover arise exclusively as rational linear combinations of the values of hypergeometric functions with rational parameters. For this class of numbers, recent breakthroughs in algebraic geometry and number theory may allow to decide equality unconditionally. In particular the preprint [13] gives a method for computing all linear relations (with algebraic coefficients) among a given collection of *E-functions* evaluated at a common algebraic point. This result can be used to compute linear relations among values of the hypergeometric function ${}_1F_1$ with rational parameters. In a similar vein, another recent preprint [21] claims to prove the Kontsevich–Zagier conjecture for the special case of 1-motives, which encompasses linear combinations of the hypergeometric function ${}_2F_1$ with rational parameters.

These recent advances, if confirmed, suggest that equality checking among the relevant classes of periods and exponential periods could be unconditionally decidable (which in turn would entail unconditional decidability results for a substantial class of instances of the decision problems on holonomic sequences considered in this paper).

B. pseudoperiods

In this paper we also encounter integrals that formally resemble univariate periods, and that arise variously as rational linear combinations of values of hypergeometric functions evaluated at real algebraic parameters, and as Beta integrals evaluated at algebraic numbers. A (univariate) *pseudoperiod* is a number that can be written as an absolutely convergent integral of the form

$$\int_\sigma \sum_{i=1}^{n} \exp\Big(\sum_{j=1}^{k} \alpha_{i,j} \operatorname{Log} f_{i,j}(x)\Big)\, dx \qquad (n, k \in \mathbb{N}),$$

where the domain $\sigma \subseteq \mathbb{R}$ is a semialgebraic set, the functions $f_{i,j}$ are real-valued, non-zero algebraic functions on σ excluding a finite number of points (the poles of f in σ), $\alpha_{i,j} \in \overline{\mathbb{Q}}$, and Log is the principal (or another, fixed) branch of the complex logarithm.

The above integral is well-defined, as the integrand is measurable[4], and so it is (Lebesgue) integrable as the integral is absolutely convergent. Since σ is a finite union of points and open intervals, we lose no generality assuming that σ is an interval with algebraic endpoints, and that moreover it does not contain any poles or zeros of the $f_{i,j}$.

The formal resemblance to periods is evidenced by defining $f(x)^\alpha := \exp(\alpha \operatorname{Log} f(x))$, whence the integrand becomes $\sum_{i=1}^n \prod_{j=1} f_{i,j}(x)^{\alpha_k}$. When all the $\alpha_{i,j}$ are rational, the pseudoperiod is readily seen to be a *bona fide* univariate period. Conversely, any univariate period is a pseudoperiod.

We remark that there is no particular reason to restrict the definition of pseudoperiods to univariate functions, in which case any period would likewise be a pseudoperiod. In this paper we only encounter univariate entities, so we omit discussion of a more general definition.

The *Pseudoperiod Equality Problem* asks, given two finite products of pseudoperiods, whether or not they are equal. For instances in which the pseudoperiods involved in the products are periods (so the products themselves are periods), Conjecture 3.1 would imply the decidability of this problem.

C. Pfaffian functions and Gabrielov-Vorobjov oracles

We conclude this section with a brief model-theoretic discussion of periods and related expressions, along with the attendant decision problems. We note that our periods, exponential periods, and pseudoperiods can also be expressed as values of univariate Pfaffian functions evaluated at rational arguments, and in turn all the decision problems considered in this paper become decidable subject to classical Gabrielov-Vorobjov oracles.

Khovanskii introduced Pfaffian functions in [26], [27]. Let U be an open domain in $\mathbb{R}$. A *Pfaffian chain* in U is a sequence of complex analytic functions $\phi_1, \phi_2, \ldots, \phi_r \colon U \to \mathbb{C}$ satisfying Pfaffian differential equations

$$\phi_j'(x) = P_j(x, \phi_1, \ldots, \phi_j) \quad \text{for } j = 1, \ldots, r$$

where each $P_j \in \mathbb{Z}[i][x, y_1, \ldots, y_j]$ is a polynomial with coefficients among the Gaussian integers. A function $\phi \colon U \to \mathbb{C}$ is called a (univariate) *Pfaffian function* if there exists a Pfaffian chain $\phi_1, \phi_2, \ldots, \phi_r$ and a polynomial $P \in \mathbb{Z}[i][x, y_1, \ldots, y_r]$ such that $\phi(x) = P(x, \phi_1, \ldots, \phi_r)$.

In [15], Gabrielov and Vorobjov introduce the concept of an oracle for deciding the consistency of a system of Pfaffian constraints.[5] Subject to such an oracle, they provide an algorithm for the smooth stratification of semi-Pfaffian sets. This notion of a *Gabrielov-Vorobjov oracle* has since been reused in a wide range of contexts (see, e.g., [45], [16], [31], [32], [33], [5]).

It is straightforward to show that any univariate period, exponential period, or pseudoperiod α can be written as $\alpha = \lim_{x\to 1^-} f(x)$, where $f : (0,1) \to \mathbb{C}$ is a univariate Pfaffian function. In fact, one can write $f(x) = \int_0^x g(y)\, dy$, where g itself is also Pfaffian on $(0,1)$.

Suppose one wishes to determine whether two given products of pseudoperiods (or [exponential] periods) $\alpha_1 \cdots \alpha_j$ and $\beta_1 \cdots \beta_j$ are equal, in other words solving a particular instance of the Pseudoperiod Equality Problem. Write each α_i as $\lim_{x\to 1^-} f_i(x)$, with $f_i(x) = \int_0^x g_i(y)\, dy$, and each β_i as $\lim_{x\to 1^-} p_i(x)$, with $p_i(x) = \int_0^x q_i(y)\, dy$, where all the functions in play are Pfaffian. Observe that each $F_i(x) := f_i(x) + \int_x^1 g_i(y)\, dy$ is Pfaffian with domain $(0,1)$, and of constant value $F_i(x) = \alpha_i$. Likewise one can write $P_i(x) = \beta_j$ for Pfaffian P_i defined over $(0,1)$. The question of whether $\alpha_1 \cdots \alpha_j = \beta_1 \cdots \beta_j$ therefore boils down to asking whether there is some $x \in (0,1)$ such that $F_1(x) \cdots F_j(x) = P_1(x) \cdots P_k(x)$, for which a Gabrielov-Vorobjov oracle readily provides an answer, since Pfaffian functions are closed under

[4]For a real-valued algebraic function f, $\operatorname{Log} f(x)$ is defined and is continuous at all but finitely many points (the poles and zeros of f), and is thus measurable. Since $\exp(\sum \alpha_i \operatorname{Log} f_i(x)) = \prod \exp(\alpha_i \operatorname{Log} f_i(x))$, and $\exp(\alpha_i \operatorname{Log} f_i(x))$ is measurable, the claim follows.

[5]Gabrielov and Vorobjov consider both real Pfaffian and complex Pfaffian functions in their paper. For real Pfaffian functions, a constraint system consists of a Boolean combination of inequalities among Pfaffian functions; the system is *consistent* if there exist values of the variables within the relevant functions' domains for which the constraint system is satisfied. In the case of complex Pfaffian functions, only Boolean combinations of equalities and disequalities are allowed.

products and we are therefore indeed dealing with a (rather simple) Pfaffian constraint system.

4. The Hypergeometric Inequality Problem reduces to the Pseudoperiod Equality Problem

First-order holonomic sequences are called *hypergeometric sequences*. In this section we consider the *Hypergeometric Inequality Problem*: given two hypergeometric sequences $\langle u_n \rangle_{n=-1}^{\infty}$ and $\langle v_n \rangle_{n=-1}^{\infty}$ determine whether $u_n \le v_n$ for each $n \in \{-1, 0, \ldots\}$. The main result of this section is the following theorem, which follows from Lemma 4.5 and Proposition 4.7.

Theorem 4.1. *The Hypergeometric Inequality Problem reduces to the Pseudoperiod Equality Problem.*

A. preliminaries for hypergeometric sequences

We collect together decidability results for hypergeometric sequences.

Let $\langle u_n \rangle_n$ be a sequence satisfying the second-order relation (1). Note that if g_2 is identically 0, then $\langle u_n \rangle_n$ consists of the interleaving of two hypergeometric sequences. At order one, both the Minimality and the Positivity Problems are algorithmically trivial: indeed, the positivity of a hypergeometric sequence is readily determined by inspecting the polynomial coefficients of its defining recurrence, together with the sign of the first few values of the sequence; and since the solution set of a hypergeometric recurrence is a one-dimensional vector space, such recurrences cannot possibly admit minimal sequences. Similarly, if g_1 is identically 0, then positivity and minimality of $\langle u_n \rangle_n$ likewise become trivial. Thus we have the following.

Lemma 4.2. *The Hypergeometric Minimality and Hypergeometric Positivity Problems are decidable.*

As an intermediate step in our proof of Theorem 4.1, we introduce the *Hypergeometric Threshold Problem*: given a hypergeometric sequence $\langle w_n \rangle_n^{\infty}$ and a real-algebraic constant θ (the threshold), determine whether $w_n \le \theta$ for each n. Clearly the Hypergeometric Positivity Problem is a specialisation of the Hypergeometric Threshold Problem with threshold $\theta = 0$.

The *Skolem Problem* is a decision problem that has garnered much attention from researchers in both number theory and theoretical computer science. Given a holonomic sequence $\langle u_n \rangle_n$, the Skolem Problem asks whether there is an index n such that $u_n = 0$. The decidability of the Skolem Problem has been settled for C-recurrences at low orders (see [41], [43] and references therein). Much less is known about the decidability of the Skolem Problem for general holonomic sequences; however, our restricted focus on hypergeometric sequences gives the following classical result (considered folklore).

Lemma 4.3. *The Hypergeometric Skolem Problem is decidable.*

We say an infinite product $\prod_{k=1}^{\infty} r(k)$ *converges* if the sequence of partial products converges to a finite non-zero limit (otherwise the product is said to diverge). Recall the following classical theorem [58], [7] where Γ is Euler's Gamma function (see [3]) defined on the domain $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Theorem 4.4. *Consider the rational function*

$$r(k) := \frac{c(k+\varphi_1)\cdots(k+\varphi_m)}{(k+\psi_1)\cdots(k+\psi_{m'})}$$

where we suppose that each $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_{m'}$ is a complex number that is neither zero nor a negative integer. The infinite product $\prod_{k=1}^{\infty} r(k)$ converges to a finite non-zero limit only if $c = 1$, $m = m'$, and $\sum_j \varphi_j = \sum_j \psi_j$. Further, the value of the limit is given by

$$\prod_{k=0}^{\infty} r(k) = \prod_{j=1}^{m} \frac{\Gamma(\psi_j)}{\Gamma(\varphi_j)}.$$

Let $\langle w_n \rangle_{n=-1}^{\infty}$ be a hypergeometric sequence given by $w_n := w_{-1} \prod_{k=0}^{n} r(k)$ with $w_{-1} \neq 0$ and $r(k)$ as above. We deduce that $\langle w_n \rangle_n$ converges to a finite non-zero limit only if $r(k)$ satisfies the conditions given in Theorem 4.4.

B. reduction argument

Lemma 4.5. *The Hypergeometric Inequality Problem reduces to the Hypergeometric Threshold Problem.*

Proof. Let $\langle \hat{u}_n \rangle_n$ and $\langle \tilde{u}_n \rangle_n$ be hypergeometric sequences associated with the respective rational functions $\hat{r}(n)$ and $\tilde{r}(n)$. We can assume that $\hat{r}(n)$ and $\tilde{r}(n)$ are constant in sign by shifting the sequence as appropriate. By Lemma 4.3, one can decide whether there exists an $n \in \{-1, 0, \ldots\}$ such that $\tilde{u}_n = 0$. If such an n exists, the Hypergeometric Inequality Problem reduces to determining the Positivity Problem for $\langle \hat{u}_n \rangle_n$ (which is trivial by Lemma 4.2) with a number of comparisons between initial values determined by the shift. Thus we can assume there is no $n \in \{-1, 0, \ldots\}$ for which $\tilde{u}_n = 0$.

Let $\langle w_n \rangle_n$ be the sequence with terms given by $w_n := \operatorname{sign}(\tilde{u}_n)\hat{u}_n/\tilde{u}_n$. The sequence $\langle w_n \rangle_n$ is well-defined and satisfies the recurrence $w_n = r(n) w_{n-1}$ where $r(n) := \operatorname{sign}(\tilde{r})\hat{r}(n)/\tilde{r}(n)$. Thus $\langle w_n \rangle_n$ is a hypergeometric sequence. It is clear that $\hat{u}_n \le \tilde{u}_n$ for each n if and only if $w_n \le \operatorname{sign}(\tilde{r})$. There are two cases to consider: either $\operatorname{sign}(\tilde{r})$ is alternating or constant. If $\operatorname{sign}(\tilde{r})$ is alternating then one considers two instances of the Hypergeometric Threshold Problem as it is sufficient to determine whether $w_{2n} \le -\operatorname{sign}(\tilde{u}_{-1})$ and $w_{2n-1} \le \operatorname{sign}(\tilde{u}_{-1})$ for each n. Thus in both cases the Hypergeometric Inequality Problem reduces to instances of the Hypergeometric Threshold Problem. □

Let us list our ongoing assumptions (which we make without loss of generality):

1) Given an instance of the Hypergeometric Threshold Problem, we shall assume that the threshold is non-zero (Lemma 4.2).
2) Given a hypergeometric sequence $\langle w_n \rangle_n$ we shall assume that $\operatorname{sign}(w_n)$ is constant (Lemma 4.5) and that there is no such $n \in \mathbb{N}$ for which $w_n = 0$ (Lemma 4.3).

We add two further assumptions to our list:

3) Given a hypergeometric sequence $\langle w_n \rangle_n$ we shall assume that the associated rational function is of the form
$$r(n) = c\frac{(n+\varphi_1)\cdots(n+\varphi_m)}{(n+\psi_1)\cdots(n+\psi_{m'})}$$
such that $c > 0$ and $\varphi_j, \psi_j \in \overline{\mathbb{Q}}$. With a suitable shift, one can assume that for each j we have $\mathrm{Re}(\varphi_j), \mathrm{Re}(\psi_j) > 0$.
4) Subject to a suitable shift, we assume that both $\langle r(n) \rangle_n$ and $\langle r(n) - 1 \rangle_n$ are monotone and have constant sign.

Lemma 4.6. *The Hypergeometric Threshold Problem reduces to the problem of determining equality between a real-algebraic number and the limit of a hypergeometric sequence.*

Proof. We have $w_n - w_{n-1} = (r(n)-1)w_{n-1}$. By assumption $\langle r(n) - 1 \rangle_n$ and $\langle w_n \rangle_n$ have constant sign, so we deduce that $\langle w_n \rangle_n$ is monotone. If $\langle w_n \rangle_n$ diverges to $\pm\infty$ then one easily determines the Hypergeometric Threshold Problem. Indeed, if $\langle w_n \rangle_n$ diverges to $-\infty$ then it suffices to check whether $w_{-1} \le \theta$. Thus we can safely assume that the limit of $\langle w_n \rangle_n$ exists and is finite.

Assume that $\lim_{n\to\infty} w_n = \ell \in \mathbb{R}$. Let θ, the threshold, be a real-algebraic constant. Then one needs to determine whether: $\ell > \theta$, in which case one can detect that eventually $w_n > \theta$; $\ell < \theta$, in which case it suffices to check whether $w_{-1} \le \theta$ as $\langle w_n \rangle_n$ is monotone; or $\ell = \theta$. Thus the Hypergeometric Inequality Problem is decidable subject to an oracle that can determine whether $\ell = \theta$. ☐

Proposition 4.7. *The Hypergeometric Threshold Problem reduces to the Pseudoperiod Equality Problem.*

Proof. By Lemma 4.6, the Hypergeometric Threshold Problem is decidable subject to determining whether $\lim_{n\to\infty} w_n = \ell$ is equal to a given threshold θ (as above). We freely assume that $\theta \neq 0$ by Assumption 1. This in turn means one can assume that $\ell \neq 0$. Thus the associated rational function $r(n)$ satisfies the necessary assumptions in Theorem 4.4 and, additionally, $\ell = \theta$ if and only if $\Gamma(\psi_1)\cdots\Gamma(\psi_k) = \theta \cdot \Gamma(\varphi_1)\cdots\Gamma(\varphi_k)$.

For $\xi, \nu \in \mathbb{C}$ with $\mathrm{Re}(\xi), \mathrm{Re}(\nu) > 0$, the *beta function* (see [3]) is given by $B(\xi,\nu) = \int_0^1 t^\xi (1-t)^\nu \, dt$, which is a pseudoperiod when ξ and ν are algebraic. Further, we have the following identity $\Gamma(\xi)\Gamma(\nu) = B(\xi,\nu)\Gamma(\xi+\nu)$, which we (repeatedly) apply to the products $\Gamma(\psi_1)\cdots\Gamma(\psi_k)$ and $\Gamma(\varphi_1)\cdots\Gamma(\varphi_k)$. For the former product we have
$$\Gamma(\psi_1)\cdots\Gamma(\psi_k) = \Gamma(\textstyle\sum_{i=1}^k \psi_i) \displaystyle\prod_{i=1}^{k-1} B(\textstyle\sum_{j\le i} \psi_j, \psi_{i+1}),$$
and we obtain an analogous expression for the latter product. Since $\sum_{i=1}^k \psi_i = \sum_{i=1}^k \varphi_i$, the problem of determining whether $\Gamma(\psi_1)\cdots\Gamma(\psi_k)$ and $\theta \cdot \Gamma(\varphi_1)\cdots\Gamma(\varphi_k)$ are equal reduces to determining whether
$$\prod_{i=1}^{k-1} B(\textstyle\sum_{j\le i} \psi_j, \psi_{i+1}) = \theta \displaystyle\prod_{i=1}^{k-1} B(\textstyle\sum_{j\le i} \varphi_j, \varphi_{i+1}). \qquad (7)$$
Consequently, determining whether $\ell = \theta$ reduces to an instance of the Pseudoperiod Equality Problem. This concludes the proof. ☐

Remark 4.8. If the numbers φ_i, ψ_i in (7) are rational, then (7) is an instance of the Period Equality Problem. Thus if one assumes that Conjecture 3.1, for such instances the Hypergeometric Inequality Problem is decidable.

For suitable rational inputs φ_i, $\psi_i \in \mathbb{Q}$, Rohrlich predicted that
$$\frac{\Gamma(\psi_1)\cdots\Gamma(\psi_k)}{\Gamma(\varphi_1)\cdots\Gamma(\varphi_m)} \in \overline{\mathbb{Q}}$$
can only occur as a consequence of the *functional relations* (translation, reflection, and multiplication) of the Gamma function (see, e.g., [55, Conjecture 21] for more information). We remark that, similar to Conjecture 3.1, Rohrlich's conjecture implies there is a semi-algorithm that can decide the Hypergeometric Inequality Problem with rational inputs by enumerating all applications of the functional relations.

5. Minimality for Degree-1 Holonomic Sequences

Let us state the main result of this section.

Theorem 5.1. *For second-order degree-*1 *holonomic sequences, the Minimality Problem reduces to the Equality Problems for pseudoperiods and exponential periods.*

Corollary 5.2. *For second-order degree-*1 *holonomic sequences, the Positivity Problem reduces to the Equality Problems for pseudoperiods and exponential periods.*

Proof of Corollary 5.2. We note that one cannot use the method in Section 2 since the degrees of the coefficients are not preserved in the reduction. However, one can reduce the Positivity Problem to the Minimality Problem in this setting (see Corollary C.5). The result follows from Theorem 5.1. ☐

Theorem 5.1 is a consequence of Corollary 5.5 and Theorem 5.7. To this end, let us parametrise the Minimality Problem as follows.

Problem 5.3 ($\mathsf{Minimal}(j,k,\ell)$)**.** Given a solution $\langle v_n \rangle_{n=-1}^\infty$ to (1) with $\deg(g_3) = j, \deg(g_2) = k$, and $\deg(g_1) = \ell$, determine whether $\langle v_n \rangle_n$ is minimal.

The cases where any of the polynomial coefficients are identically 0 are considered in Section 4 hence our focus on $j,k,\ell \in \{0,1\}$.

A. *interreductions for* $\mathsf{Minimal}(j,k,\ell)$

Consider problem $\mathsf{Minimal}(0,0,0)$: determine whether a holonomic sequence that solves a second-order C-finite recurrence is a minimal solution. Since this problem is a special case of $\mathsf{Minimal}(1,1,1)$ (multiply each of the coefficients by $(n+1)$) we make no further mention of $\mathsf{Minimal}(0,0,0)$ in the sequel. It therefore appears that we need to consider eight instances of $\mathsf{Minimal}(j,k,\ell)$. We first reduce the number of problems to five by establishing interreductions between instances of $\mathsf{Minimal}(j,k,\ell)$. Next we employ minimality-preserving transformations to obtain canonical instances of each of the remaining problems (Corollary 5.5). In the sequel we show that these canonical instances reduce to checking whether a pseudoperiod or an exponential period vanishes. The different cases are listed in Theorem 5.7.

For the remaining problem instances it is useful to establish the following conventions. In each instance $\mathsf{Minimal}(j,k,\ell)$ we consider a recurrence relation of the form

$$(\alpha_1 n+\alpha_0)u_n = (\beta_1 n+\beta_0)u_{n-1} + (\gamma_1 n+\gamma_0)u_{n-2}. \quad (8)$$

In the case that $\deg(\alpha_1 n+\alpha_0)=0$ we understand that $\alpha_1=0$ (and adopt a similar convention for the other coefficients). We shall always assume that the values α_0, β_0, and γ_0 are non-zero in accordance with the assumption that the polynomial coefficients do not vanish on non-negative integers. We also introduce the notation $\alpha := \alpha_0/\alpha_1$.

For recurrence (8) with $\alpha_1 \neq 0$, we define the associated *characteristic polynomial* as $\alpha_1 x^2 - \beta_1 x - \gamma_1$, and refer to its roots as the associated *characteristic roots*.

The proof of the following proposition is given in Appendix D.

Proposition 5.4.

1) $\mathsf{Minimal}(0,k,1)$ *and* $\mathsf{Minimal}(1,k,0)$ *are interreducible.*
2) $\mathsf{Minimal}(1,0,0)$ *reduces to* $\mathsf{Minimal}(1,1,1)$.
3) $\mathsf{Minimal}(1,1,1)$ *reduces to the Minimality Problem for a recurrence of the form*

$$u_n = \frac{\beta_1 n+\beta_0}{n+\alpha}u_{n-1} + \frac{\gamma_1 n+\gamma_0}{n+\alpha}u_{n-2}, \quad (9)$$

where $\alpha, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{Q}$, $\beta_1 > 0$, *and* $|\gamma_1| = \beta_1$. *The characteristic roots associated to the recurrence are* $(\beta_1 \pm \sqrt{\beta_1^2+4\gamma_1})/2$. *If* $\beta_1^2+4\gamma_1=0$, *then we have a further reduction to the Minimality Problem for solutions to*

$$u_n = \frac{2n+\beta_0}{n+\alpha}u_{n-1} - \frac{n+\gamma_0}{n+\alpha}u_{n-2}, \quad (10)$$

where $\alpha, \beta_0, \gamma_0 \in \mathbb{Q}$ *and the recurrence has a single repeated characteristic root* 1.

4) $\mathsf{Minimal}(1,0,1)$ *reduces to the Minimality Problem for solutions to a recurrence of the form*[6]

$$u_n = \frac{\beta_0}{n+\alpha}u_{n-1} + \frac{n+\gamma_0}{n+\alpha}u_{n-2}, \quad (11)$$

where $\alpha, \gamma_0 \in \mathbb{Q}$ *and* $\beta_0 \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$. *The characteristic roots associated to the recurrence are* ± 1.

Let λ and μ be the roots of the associated characteristic polynomial such that $|\lambda| \le |\mu|$. In recurrences (9)–(11) γ_1 is not zero so we have $\lambda, \mu \neq 0$. Further, by setting $\mu = 1$ for (11), we have that $\mu > 0$ for the associated recurrences (9)–(11), as the coefficient $\beta_1 > 0$ in the first two.

We shall treat recurrences (9) and (10) as distinct cases in the sequel. That is to say, in the former we shall always assume that $\beta_1^2+4\gamma_1 \neq 0$ (so that the characteristic roots are distinct).

The following corollary is an immediate application of Proposition 5.4.

[6]Notice that sequences satisfying (11) are not necessarily holonomic as β_0 can be a real algebraic number.

Corollary 5.5. *For* $j,k,\ell \in \{0,1\}$, *decidability of problem* $\mathsf{Minimal}(j,k,\ell)$ *reduces to proving decidability of* $\mathsf{Minimal}(0,1,0)$, $\mathsf{Minimal}(0,1,1)$, *and decidability of the Minimality Problem for solutions to recurrences* (9)–(11).

Recall from Corollary 2.11 that it is decidable whether a second-order degree-1 recurrence relation admits minimal solutions. The next lemma gives necessary and sufficient conditions for the relevant recurrences to admit minimal solutions. The proof is given in Appendix E. In particular, in the sequel we shall assume that the characteristic roots of a recurrence relation are real-valued.

Lemma 5.6.

1) *A recurrence* $u_n = (\beta_1 n+\beta_0)u_{n-1} + (\gamma_1 n+\gamma_0)u_{n-2}$ *with* $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{Q}$, *with* $\beta_0, \beta_1, \gamma_0 \neq 0$, *admits minimal solutions.*
2) *A recurrence of the form* (9) *or* (11) *admits minimal solutions if and only if the associated characteristic roots are real.*
3) *A recurrence of the form* (10) *admits minimal solutions if and only if* $\beta_0 - \alpha - \gamma_0 \ge 0$.

To prove Theorem 5.1, in light of Corollary 5.2, the following theorem suffices.

Theorem 5.7.

1) $\mathsf{Minimal}(0,1,0)$ *reduces to checking whether an exponential period vanishes.*
2) $\mathsf{Minimal}(0,1,1)$ *reduces to checking whether an exponential period vanishes.*
3) *The Minimality Problem for recurrences* (9) *and* (11) *reduce to the Pseudoperiod Equality Problem.*
4) *The Minimality Problem for recurrences* (10) *reduces to checking whether an exponential period vanishes.*

The rest of this section is devoted to proving the above theorem. We approach the proof of Theorem 5.1 via work by Lorentzen and Waadeland in [37, §VI.4.1]. They consider low-degree polynomial continued fractions and express their limits as quotients of *hypergeometric functions*. These functions admit, after suitable transformations, integral representations that involve pseudoperiods or exponential periods. Let us first define these entities and recall some of their properties from [3].

B. preliminaries on hypergeometric functions

A series $\sum c_k x^k$ is called *hypergeometric* if the sequence $\langle c_n \rangle_n$ is hypergeometric. It can be shown (see [3]) that a hypergeometric series can be written as follows

$$\sum_{k=0}^{\infty} c_k x^k = c_0 \sum_{k=0}^{\infty} \frac{(\varphi_1)_k \cdots (\varphi_j)_k}{(\psi_1)_k \cdots (\psi_\ell)_k} \frac{x^k}{k!},$$

where, for $\rho \in \mathbb{C}$, the *(rising) Pochhammer symbol* $(\rho)_n$ is defined as $(\rho)_0 = 1$, and $(\rho)_n = \prod_{j=0}^{n-1}(\rho+j)$ for $n \ge 1$. Here the parameters ψ_m are not negative integers or zero for otherwise the denominator would vanish for some k. The series terminates if any of the parameters φ_m is zero or a negative integer. We let ${}_jF_\ell(\varphi_1,\ldots,\varphi_j;\psi_1,\ldots,\psi_\ell;x)$ denote the series on the right-hand side.

A hypergeometric series ${}_jF_\ell(\varphi_1,\ldots,\varphi_j;\psi_1,\ldots,\psi_\ell;x)$ converges absolutely for all x if $j \le \ell$ and for $|x| < 1$ if $j = \ell+1$. It diverges for all $x \neq 0$ if $j > \ell+1$ and the series does not terminate. Furthermore, in the case that $j = \ell+1$, the series with $|x| = 1$ converges absolutely if $\mathrm{Re}(\sum\psi_i - \sum\varphi_i) > 0$. The series converges conditionally if $x \neq 1$ and $0 \ge \mathrm{Re}(\sum\psi_i - \sum\varphi_i) > -1$. The series diverges if $\mathrm{Re}(\sum\psi_i - \sum\varphi_i) \le -1$.

Abusing notation, we let ${}_jF_\ell(\varphi_1,\ldots,\varphi_j;\psi_1,\ldots,\psi_\ell;x)$ denote the analytic function defined by the corresponding series in its radius of convergence (assuming it has a positive radius of convergence), and elsewhere by analytic continuation.

Example 5.8. Of special interest to us is the *Gauss hypergeometric function* ${}_2F_1(a,b;c;x)$, with $a,b,c \in \mathbb{C}$, c neither zero nor a negative integer. It is a single-valued function on the cut complex plane $\mathbb{C}\setminus[1,\infty)$ (cf. [37, §VI.1]).

Another function of interest is the *confluent hypergeometric function (of the first kind)* ${}_1F_1(a;b;x)$, b neither zero nor a negative integer. The series defines an entire function.

For $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, Euler gave the following integral representation: for all $x \in \mathbb{C}\setminus[1,\infty)$, ${}_2F_1(a,b;c;x)$ is equal to

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-xt)^a}\,dt, \tag{12}$$

where t^b, $(1-t)^{c-b-1}$, and $(1-xt)^{-a}$ have their principal values. Here the path of integration is understood to be along the real line between 0 and 1. We observe that for a, b, c, $x \in \overline{\mathbb{Q}}$, the integral in (12) is a pseudoperiod.

C. proof of Theorem 5.7

Here in we only include the proof of Theorem 5.7(3). The proofs of the remaining items are given in the appendices (found in Appendix E) follow similarly.

Proof of Theorem 5.7(3). Consider recurrence (9). Recall μ and λ are the roots of the characteristic polynomial of recurrence with $\mu > |\lambda| > 0$. Let $\langle u_n\rangle_n$ be a minimal solution to the recurrence. By Theorem 2.5, $-u_0/u_{-1} = \mathrm{K}(a_n/b_n)$. Then $\mathrm{K}(a_n/b_n)$ is equivalent to the continued fraction

$$\frac{1}{\alpha\delta^2}\operatorname*{K}_{n=1}^{\infty}\frac{(\gamma_1 n+\gamma_0)(n+\alpha-1)\delta^2}{(\beta_1 n+\beta_0)\delta}$$

by applying Theorem 2.2 with $\mathfrak{r}_0 = 1$, $\mathfrak{r}_n = \delta(n+\alpha)$ for $n \in \mathbb{N}$, and δ a non-zero real number. Setting $\delta = 1/\mu$, $x = \lambda/\mu$, and either

$$a = \frac{\lambda^2\alpha - \lambda\beta_0 - \gamma_0}{\lambda(\lambda-\mu)},\ b = \alpha - 1,\ c = a + \frac{\gamma_0}{\gamma_1};\quad \text{or}$$
$$a = \frac{\mu^2\alpha - \mu\beta_0 - \gamma_0}{\mu(\lambda-\mu)} + 1,\ b = \frac{\gamma_0}{\gamma_1},\ c = a + \alpha - 1$$

then, by substitution, the above continued fraction can be written as

$$\frac{\mu^2}{\alpha}\operatorname*{K}_{n=1}^{\infty}\frac{-(c-a+n)(b+n)x}{c+n+(b-a+1+n)x}. \tag{13}$$

Here we note that $\gamma_1 = -\lambda\mu$ and $\beta_1 = \lambda+\mu$. Notice here that $|x| < 1$ by assumption. Furthermore, by shifting the sequence appropriately, the parameter a does not change, but b and c are shifted by an integer value. Hence we may assume that c is a positive number. Finally, $a,b,c,x \in \mathbb{Q}(\mu)$ and so are real algebraic numbers. By [37, §VI, Theorem 4(A)]

$$-\frac{\alpha}{\mu^2}\frac{u_0}{u_{-1}} = \frac{c\,{}_2F_1(a,b;c;x)}{{}_2F_1(a,b+1;c+1;x)} - \frac{\beta_0}{\mu}. \tag{14}$$

Let us first show that the equality problem is decidable if a, b, $c-a$, or $c-b$ is a negative integer, or if either $a = 0$, or $c - b = 0$. If either $a = 0$, or a or b is a negative integer, then both the hypergeometric series terminate. Consequently the equality reduces to the equality problem for algebraic numbers, which is decidable. Moreover, we have Euler's transformation (cf. [3, Eq. 2.2.7]):

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x),$$

which implies that if $c-b = 0$ or one of $c-a$ and $c-b$ is a negative integer then again the series terminates, and the problem reduces to checking equality between algebraic numbers. Thus for the remainder of the proof we assume that $a, b, c-a, c-b$ are not negative integers and, in addition, that a and $c-b$ are non-zero.

Consider the sequence $\langle P_n(x)\rangle_n$ defined by

$$P_{2n}(x) := {}_2F_1(a+n,b+n;c+2n;x);\quad \text{and}$$
$$P_{2n+1}(x) := {}_2F_1(a+n,b+n+1;c+2n+1;x).$$

Let $\langle s_n\rangle_{n=1}^{\infty}$ be the sequence of linear fractional transformations given by by $s_n(w) := a_n x/(1+w)$ such that

$$a_{2n+1} = \frac{-(a+n)(c-b+n)}{(c+2n)(c+2n+1)},\ a_{2n} = \frac{-(b+n)(c-a+n)}{(c+2n-1)(c+2n)}.$$

It can be shown that

$$\frac{P_0(x)}{P_1(x)} = s_1\circ\cdots\circ s_n\left(\frac{P_n(x)}{P_{n+1}(x)}\right)$$

(see [37, § VI.1]). Under the aforementioned assumptions we have $a_i \neq 0$ for each $i = 1,\ldots,n$, and so the composition of the s_i is an invertible linear fractional transformation. It follows that

$$\frac{P_0(x)}{P_1(x)} = \frac{X_nP_{n+1}(x) + Y_nP_n(x)}{Z_nP_{n+1}(x) + W_nP_n(x)}, \tag{15}$$

where the sequences $\langle W_n\rangle_n, \langle X_n\rangle_n, \langle Y_n\rangle_n, \langle Z_n\rangle_n$ are over $\mathbb{Q}(a,b,c,x)$ and satisfy $X_nW_n - Y_nZ_n \neq 0$.

Fix $N \in \mathbb{N}$ even and sufficiently large such that $c + 2N > b + N$. Substituting (15) into (14) with $n = N$ and rearranging, we obtain the equation $\mathfrak{a}_N P_{N+1}(x) = \mathfrak{b}_N P_N(x)$, where

$$\mathfrak{a}_N := X_N - Z_N\frac{\beta_0\mu - \alpha u_0/u_{-1}}{c\mu^2}\quad \text{and}$$
$$\mathfrak{b}_N := \frac{\beta_0\mu - \alpha u_0/u_{-1}}{c\mu^2}W_N - Y_N.$$

Observe now that Euler's integral representation (12) holds for both P_{N+1} and P_N. By linearity of the integral, we see that the Minimality Problem for recurrences (9) reduces to checking whether the integral

$$\int_0^1 \frac{t^{b+N-1}(1-t)^{c-b+N-1}}{(1-xt)^{a+N}}\left(\mathfrak{a}_N \tfrac{(c+2N)}{b+N} t - \mathfrak{b}_N\right) dt$$

vanishes. This is a pseudoperiod by inspection, so we are done for this part.

Consider then recurrence (11). Similar to the above, for a minimal solution $\langle u_n \rangle_n$, $-\alpha u_0/u_{-1}$ is equal to (13) when one sets $x=-1$, and either

$$a = \frac{1}{2}(\alpha+\beta_0-\gamma_0),\ b=\alpha-1,\ c=\frac{1}{2}(\alpha+\beta_0+\gamma_0); \quad \text{or}$$
$$a=\frac{1}{2}(\beta_0+\gamma_0-\alpha)+1,\ b=\gamma_0,\ c=\frac{1}{2}(\beta_0+\gamma_0+\alpha).$$

As $c+a-b-1=\beta_0>0$ and the parameters are real-valued, by [37, §VI, Thm. 4(A)] the continued fraction converges to (14). The rest of the proof is the same. □

For Theorem 5.7(1) we show that the equality problem reduces to checking whether ${}_0F_1(;a;x)/{}_0F_1(;a+1;x)$, where $a,x\in\mathbb{Q}$, is equal to an algebraic number. Similarly, for Theorem 5.7(2) the problem reduces to checking whether ${}_1F_1(a;b;x)/{}_1F_1(a+1;b+1;x)$ is equal to an algebraic number. Here again $a,x\in\mathbb{Q}$. Finally for Theorem 5.7(4) we show that the problem reduces to checking whether $U(a,b,x)/U(a+1,b+1,x)$ is equal to an algebraic number. Here again $a,b,x\in\mathbb{Q}$, and U is defined for all $a,b,x\in\mathbb{C}$, $\mathrm{Re}(a),\mathrm{Re}(x)>0$ as

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt}t^{a-1}(1+t)^{b-a-1}\,dt.$$

The definition of U immediately gives the desired result after rearranging (and shifting the sequence appropriately).

To complete the proof bring forth the integral representations (see [1, Eq. 9.1.69] and [3, Eq. 4.7.5], and [1, Eq. 13.2.1], respectively):

$${}_0F_1(;s+1;z)=\frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+1/2)}\int_{-1}^1 \mathrm{e}^{-2\sqrt{z}t}(1-t^2)^{s-1/2}\,dt,$$
$${}_1F_1(a;b;x)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\mathrm{e}^{xt}\,dt.$$

These hold for $\mathrm{Re}(a)>-1/2$ and $\mathrm{Re}(b)>\mathrm{Re}(a)>0$, respectively. Reducing the correponding equalities to vanishing of exponential periods is then straightforward.

Appendix A
The PCF Equality Problem and the Minimality Problem are interreducible

Corollary 2.7 is a straightforward application of Pincherle's Theorem (Theorem 2.5).

Proof of Corollary 2.7. An equivalence transformation will take as an input a solution $\langle u_n\rangle_n$ of recurrence $g_3(n)u_n = g_2(n)u_{n-1}+g_1(n)u_{n-2}$ and ouput a solution $\langle v_n\rangle_n$ of recurrence $v_n=g_2(n)v_{n-1}+g_1(n)g_3(n-1)v_{n-2}$ such that $\langle u_n\rangle_n$ is a minimal solution if and only if $\langle v_n\rangle_n$ is a minimal solution.

The latter of the two recurrence relations is associated with the polynomial continued fraction $\mathrm{K}(a_n/b_n)$ with partial quotients $b_n=g_2(n)$ and $a_n=g_1(n)g_3(n-1)$ for each $n\in\mathbb{N}$. By Theorem 2.5, $\langle v_n\rangle_n$ is a minimal solution if and only if $\mathrm{K}(a_n/b_n)$ converges to the limit $-v_0/v_{-1}$. Thus if one has an oracle that can determine the value of a polynomial continued fraction, then one can determine whether $\langle v_n\rangle_n$ is a minimal solution. Since minimality is preserved by the equivalence transformation, one can determine whether $\langle u_n\rangle_n$ is a minimal solution.

Conversely, given a polynomial continued fraction $\mathrm{K}(a_n/b_n)$ and an algebraic number $\xi\in\mathbb{R}$, let us construct the holonomic sequence $\langle v_n\rangle_{n=-1}^\infty$ as follows. For for each $n\in\mathbb{N}$, let $v_n = b_nv_{n-1}+a_nv_{n-2}$ with initial conditions $v_{-1}=1$ and $v_0=-\xi$. By Theorem 2.5, sequence $\langle v_n\rangle_n$ is a minimal solution of the recurrence relation if and only if the continued fraction $\mathrm{K}(a_n/b_n)$ converges to the value $-u_0/u_{-1}=\xi$. Hence if one has an oracle that can determine whether a given holonomic sequence is a minimal solution, then one can test the value of a polynomial continued fraction. □

Appendix B
Detecting positive and dominant solutions

The goal of this section is to prove Proposition 2.16. Proposition 2.16 follows from the results in Corollary B.3 and Corollary B.8.

Broadly speaking, we describe a semi-algorithm with inputs $\langle w_n\rangle_n$. This semi-algorithm terminates in finite time for sequences that are dominant with output *'input is a positive sequence'* or *'input is not a positive sequence'*, as appropriate. The semi-algorithm is non-terminating when given a minimal solution as an input. There is not a running time for the semi-algorithm: termination depends on the distance between $-w_0/w_{-1}$ and $\mathrm{K}_{n=1}^\infty(\kappa_n/1)$.

Recall from Pincherle's Theorem that $\langle w_n\rangle_n$ is a minimal solution of recurrence (4) if and only if $-w_0/w_{-1}=f:=\mathrm{K}_{n=1}^\infty(\kappa_n/1)$. Thus if $\langle w_n\rangle_n$ is minimal we have

$$-w_m/w_{m-1}=f^{(m)}:=\mathop{\mathrm{K}}_{n=m+1}^{\infty}(\kappa_n/1)$$

for each $m\in\mathbb{N}$. In the literature the sequence $\langle f^{(m)}\rangle_{m=0}^\infty$ is the *sequence of tails of the continued fraction* $\mathrm{K}_{n=1}^\infty(\kappa_n/1)$ [37], [38].

Given a non-trivial solution sequence $\langle w_n\rangle_n$, let $t_n := -w_n/w_{n-1}$ for each $n\in\mathbb{N}$. By (3), $s_n^{-1}(t_{n-1})=t_n$ where $s_n^{-1}(w)=-1+\kappa_n/w$ is a univalent linear fractional transformation with inverse $s_n(w)=\kappa_n/(1+w)$. The sequence $\langle t_n\rangle_n$ is called a *tail sequence* for the continued fraction $\mathrm{K}(\kappa_n/1)$ [37], [38]. The aforementioned sequence $\langle f^{(n)}\rangle_n$ is one such example. A tail sequence for $\mathrm{K}(\kappa_n/1)$ is wholly determined by its initial value t_0.

The linear fractional transformation $s_n(w)=\kappa_n/(1+w)$ has two fixed points $\omega_n^\pm := \frac{1}{2}(-1\pm\sqrt{1+4\kappa_n})$. Analysis shows that ω_n^+ is an attracting fixed point of s_n, while ω_n^- is a repelling one.

By assumption, $\langle\sqrt{1+4\kappa_n}\rangle_n$ converges to a real value. We split our analysis into two cases depending on whether κ_n converges to $-1/4$. These subcases are common in the literature as some of the convergence properties of the continued fraction $\mathrm{K}(\kappa/1)$ (with $\kappa := \lim_{n\to\infty}\kappa_n$ hold for the continued fraction $\mathrm{K}(\kappa_n/1)$. In fact, the subcases of *limit hyperbolic-* and *parabolic type* are named for the classification of the limiting linear fractional transformation $s(w) = \kappa/(1+w)$.

A. limit hyperbolic type

Let us first recall an asymptotic result established by Poincaré and Perron in the restricted setting of second-order recurrence relations. Let $\langle a_n\rangle_{n=1}^\infty$ and $\langle b_n\rangle_{n=1}^\infty$ be real-valued sequences. We say that $u_n = b_n u_{n-1} + a_n u_{n-2}$ is a *Poincaré recurrence* if the limits $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ exist and are finite. The next result, initially considered by Poincaré [52] and expanded upon by Perron [46], considers Poincaré recurrences as perturbations of C-finite recurrences.

Theorem B.1 (Poincaré–Perron Theorem)**.** *Suppose that $u_n = b_n u_{n-1} + a_n u_{n-2}$ is a Poincaré recurrence and $a_n, b_n \neq 0$ for each $n \in \mathbb{N}$. Let λ and μ be the roots of the associated characteristic polynomial $x^2 - bx - a$ and suppose that $|\lambda| \neq |\mu|$. Then the above recurrence has two linearly independent solutions $\langle\hat{u}_n\rangle_n$ and $\langle\tilde{u}_n\rangle_n$ such that $\hat{u}_{n+1}/\hat{u}_n \sim \lambda$ and $\tilde{u}_{n+1}/\tilde{u}_n \sim \mu$.*

A continued fraction $\mathrm{K}(\kappa_n/1)$ is of *limit hyperbolic type* if the finite value $\kappa := \lim_{n\to\infty}\kappa_n$ satisfies $\kappa > -1/4$. In this case the sequences $\langle\omega_n^+\rangle_n$ and $\langle\omega_n^-\rangle_n$ converge to distinct limits ω^+ and ω^-, respectively. We shall assume, without loss of generality, that aforementioned eventually statements hold for each $n \in \mathbb{N}$.

The following theorem is an application of Theorem B.1 and Theorem 2.5. The connection to the convergence of tail sequences associated with a continued fraction of limit hyperbolic type is discussed in [38, Theorem 4.13].

Theorem B.2. *Suppose that $\mathrm{K}(\kappa_n/1)$ is of limit hyperbolic type. The sequence of tails $\langle f^{(n)}\rangle_n$ converges to ω^+. A tail sequence $\langle t_n\rangle_n$ with $t_0 \neq f^{(0)}$ converges to ω^-.*

Corollary B.3. *Suppose that $\mathrm{K}(\kappa_n/1)$ is of limit hyperbolic type. One can detect if a solution sequence $\langle w_n\rangle_n$ of recurrence* (4) *is positive and dominant.*

Proof. Let $\langle t_n\rangle_n$ be the tail sequence associated with a non-trivial solution $\langle w_n\rangle_n$. By Theorem B.2, a tail sequence $\langle t_n\rangle_n$ associated with a dominant solution converges to ω^- in the limit, whilst the tail sequence $\langle f^{(n)}\rangle_n$ associated with a minimal solution converges to ω^+ in the limit.

We can compute $N \in \mathbb{N}$ so that for all $n \geq N$, the two fixed points of s_n^{-1} are separated: $\omega_n^- < (\omega^- + \omega^+)/2 = -1/2 < \omega_n^+$. Therefore, a tail sequence associated with a dominant solution is eventually bounded above by $-1/2$ since ω_n^- is attracting for each n. We conclude that if a dominant sequence is negative, it is so before its tail crosses the $-1/2$ threshold. This allows us to decide positivity of dominant sequences. $\square$

B. limit parabolic type

A continued fraction $\mathrm{K}(\kappa_n/1)$ is of *limit parabolic type* if $\lim_{n\to\infty}\kappa_n = -1/4$. In this case both $\langle\omega_n^+\rangle_n$ and $\langle\omega_n^-\rangle_n$ converge to $-1/2$. Following Lemma 2.13, we shall assume that (5) holds for each $n \in \mathbb{N}$.

The limit parabolic case is subtler than the limit hyperbolic case; this is best illustrated by the following result: all tail sequences converge to the same limit (see the general case [38, Theorem 4.17]).

Theorem B.4. *Let $\mathrm{K}(\kappa_n/1)$ be a continued fraction of limit parabolic type such that* (5) *holds for each $n \in \mathbb{N}$. Each tail sequence $\langle t_n\rangle_n$ associated with $\mathrm{K}(\kappa_n/1)$ converges to $-1/2$.*

From our assumption that (5) holds for each $n \in \mathbb{N}$, we have bounds on the sequence of tails $f^{(n-1)} := \mathrm{K}_{m=n}^\infty(a_m/b_m)$ by the following generalisation of Worpitzky's Theorem (see, for example, [38, Theorem 3.30]).

Theorem B.5. *Let $\langle\kappa_n\rangle_n$ be a sequence such that* (5) *holds for $n \in \mathbb{N}$. Then $\mathrm{K}(\kappa_n/1)$ converges to a finite value f with $0 < |f| \leq 1$ and $|f^{(n)}| \leq \mathfrak{g}_n$ for each n where $\mathfrak{g}_0 = 1$ and $\mathfrak{g}_n := 1/2 + 1/(4n) + 1/(4n\log n)$ for each $n \in \mathbb{N}$.*

The next two lemmas are simple applications of Theorem B.5 The inequalities given in the proofs follow from the observation that $s_n^{-1}\colon (-\infty, 0) \to (-1, \infty)$ is an orientation preserving bijection.

Lemma B.6. *Suppose that $\mathrm{K}(\kappa_n/1)$ is of limit parabolic type such that* (5) *holds for each $n \in \mathbb{N}$. Then $f^{(n-1)} > \kappa_n/(1-\mathfrak{g}_n)$ for each $n \in \mathbb{N}$.*

Proof. Assume, for a contradiction, that there exists an $n \in \mathbb{N}$ such that $f^{(n-1)} \leq \kappa_n/(1-\mathfrak{g}_n)$. Then

$$f^{(n)} = s_n^{-1}(f^{(n-1)}) \leq s_n^{-1}(\kappa_n/(1-\mathfrak{g}_n)) = -\mathfrak{g}_n.$$

Since $f^{(n-1)} = \mathrm{K}_{m=n}^\infty(\kappa_m/1)$, by Theorem B.5 it is not possible for $f^{(n)} < -\mathfrak{g}_n$. We conclude that $f^{(n)} = -\mathfrak{g}_n$. Since $\langle f^{(n)}\rangle_n$ is a tail sequence,

$$\begin{aligned} f^{(n+1)} - f^{(n)} &= s_{n+1}^{-1}(f^{(n)}) - f^{(n)} \\ &= -((f^{(n)})^2 + f^{(n)} - \kappa_{n+1})/f^{(n)}. \end{aligned}$$

Thus when $f^{(n)} = -\mathfrak{g}_n$ we find that $f^{(n+1)} - f^{(n)} < 0$. However this conclusion also contradicts Theorem B.5. The result follows. $\square$

Lemma B.7. *Suppose that $\mathrm{K}(\kappa_n/1)$ is of limit parabolic type such that* (5) *holds for each $n \in \mathbb{N}$. Let $\langle t_n\rangle_n$ be a tail sequence such that $f^{(0)} - t_0 > 0$. Then there exists an $N \in \mathbb{N}$ such that $t_{n+N-1} \leq \kappa_{n+N}/(1-\mathfrak{g}_n)$ for each $n \in \mathbb{N}$.*

Proof. If there exists an $N \in \mathbb{N}$ such that $t_{N-1} < \kappa_N/(1-\mathfrak{g}_N)$ then, as in the proof of Lemma B.6, $t_{n+N} \leq -\mathfrak{g}_N$ for each $n \in \mathbb{N}$. Thus it is sufficient to assume, for a contradiction, that $t_{n-1} > \kappa_n/(1-\mathfrak{g}_n)$ for each $n \in \mathbb{N}$. As the derivative of s_n^{-1} is bounded from below by unity on the interval $(-\sqrt{-\kappa_n}, 0)$, we apply the Mean Value Inequality to obtain $f^{(n)} - t_n > f^{(n-1)} - t_{n-1}$. By induction it follows that

$$f^{(n)} - t_n > f^{(n-1)} - t_{n-1} > \cdots > f^{(0)} - t_0.$$

Since both $\langle f^{(n)} \rangle_n$ and $\langle t_n \rangle_n$ are tail sequences, this outcome contradicts Theorem B.4. The desired result follows. □

Corollary B.8. *Suppose that* $\mathrm{K}(\kappa_n/1)$ *is of limit parabolic type such that* (5) *holds for each* $n \in \mathbb{N}$. *One can detect if a solution sequence* $\langle w_n \rangle_n$ *of recurrence* (4) *is positive and dominant.*

Proof. Let $\langle t_n \rangle_n$ be the tail sequence associated with a non-trivial solution $\langle w_n \rangle_n$. If $\langle w_n \rangle_n$ is dominant and positive, one has $f^{(0)} - t_0 > 0$. By Lemma B.6 and Lemma B.7, there exists an $N \in \mathbb{N}$ such that for some N and all $n \in \mathbb{N}$ we have $t_{n+N-1} < \kappa_{n+N}/(1 - \mathfrak{g}_n)$. Hence, we can detect if a sequence is dominant and positive by computing its terms until its tail sequence crosses $2\kappa_N$. □

Appendix C
The Positivity Problem reduces to the Minimality Problem at low degrees

In Section 2-D we prove that the Positivity Problem reduces to the Minimality Problem. One intermediate step shows that we need only consider recurrence relations with signature $(+,-)$. However, the argument that transforms a $(-,+)$ relation into a $(+,-)$ relation does not preserve the degrees of the polynomial coefficients. This poses a problem for our decidability result in Section 5 where we restrict our consideration to degree-1 relations. In this appendix we prove that the Positivity Problem reduces to the Minimality Problem for degree-1 second-order relations.

A. convergence results for positive continued fractions

A continued fraction $\mathrm{K}(a_n/b_n)$ is said to be *positive* if $a_n > 0$ and $b_n \geq 0$ for each $n \in \mathbb{N}$. Let us recall standard results for positive continued fractions in our reduction argument. The first result concerns the monotonicity of the odd and even approximants of positive continued fractions [37], [38].

Lemma C.1. *Suppose that for each* $n \in \mathbb{N}$ *the sequences* $\langle a_n \rangle_n$ *and* $\langle b_n \rangle_n$ *are positive. Let* $\langle f_n \rangle_n$ *be the sequence of approximants associated with* $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$. *Then* $f_2 \leq f_4 \leq \cdots \leq f_{2m} \leq \cdots \leq f_{2m+1} \leq \cdots \leq f_3 \leq f_1$. *If, in addition,* $b_1 > 0$ *then the subsequences* $\langle f_{2n} \rangle_n$ *and* $\langle f_{2n+1} \rangle_n$ *converge to finite, non-negative limits.*

We recall a necessary and sufficient criterion for convergence of a positive continued fraction [38, Theorem 3.14].

Theorem C.2 (Stern–Stolz Theorem)**.** *A positive continued fraction* $\mathrm{K}(a_n/b_n)$ *converges if and only if its* Stern–Stolz series $\sum_{n=1}^{\infty} \left| b_n \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}} \right|$ *diverges to* ∞.

B. reduction argument

We shall require the following technical lemma.

Lemma C.3. *Suppose that* $\langle u_n \rangle_n$ *is a solution sequence for recurrence* (8) *with signature* $(-,+)$. *Assume that* $u_{-1} > 0$. *For even* $n \in \mathbb{N}$, $u_n > 0$ *if and only if* $f_n > -u_0/u_{-1}$. *For odd* $n \in \mathbb{N}$, $u_n > 0$ *if and only if* $f_n < -u_0/u_{-1}$.

Proof. Recall the canonical solution sequences $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$. Then $u_n = A_n u_{-1} + B_n u_0$ for each $n \in \{-1, 0, \ldots\}$. For recurrences with signature $(-,+)$, it is easy to show by induction that $B_n < 0$ for each odd $n \in \mathbb{N}$, and $B_n > 0$ for each even $n \in \mathbb{N}$. Thus for even $n \in \mathbb{N}$ we have that $u_n > 0$ if and only if $A_n/B_n + u_0/u_{-1} = f_n + u_0/u_{-1} > 0$. The case for odd $n \in \mathbb{N}$ is handled in the same fashion. □

Proposition C.4. *Suppose that* $\langle u_n \rangle_n$ *with initial values* $u_{-1}, u_0 > 0$ *is a solution sequence for recurrence* (8) *with signature* $(-,+)$ *and the associated continued fraction* $\mathrm{K}(a_n/b_n)$ *converges to a finite limit* f. *The following statements are equivalent:*

1) *the sequence* $\langle u_n \rangle_n$ *is positive,*
2) *the sequence* $\langle u_n \rangle_n$ *is minimal, and*
3) $-u_0/u_{-1} = f$.

Proof. The sequence $\langle -f_n \rangle_{n=1}^{\infty}$ is the sequence of approximants associated with $\mathrm{K}(a_n/{-b_n})$. This is a positive continued fraction and so, by Lemma C.1, the subsequences $\langle -f_{2n} \rangle_{n=1}^{\infty}$ and $\langle -f_{2n-1} \rangle_{n=1}^{\infty}$ converge to finite limits $-\ell_1$ and $-\ell_2$, respectively. By Lemma C.3, a solution sequence $\langle u_n \rangle_n$ is positive if and only if $\ell_2 \leq -u_0/u_{-1} \leq \ell_1$. The Stern–Stolz series (from Theorem C.2) associated with $\mathrm{K}(a_n/-b_n)$ diverges due to our assumption that each of the coefficients in (8) is a polynomial with degree in $\{0,1\}$. We conclude that $\ell_1 = \ell_2$. Thus $\langle u_n \rangle_n$ is positive if and only if $-u_0/u_{-1}$ is equal to $f = \ell_1 = \ell_2$. From Theorem 2.5, a solution sequence $\langle u_n \rangle_n$ is minimal if and only if $-u_0/u_{-1}$ is the value of the continued fraction $\mathrm{K}(a_n/b_n)$. □

We obtain the following corollary by combining the results in Proposition C.4 and Proposition 2.15

Corollary C.5. *For degree-1 second-order holonomic sequences the Positivity Problem reduces to the Minimality Problem.*

Appendix D
Interreductions Between Degree-1 Holonomic Sequences

Recall that the recurrence of interest is $(\alpha_1 n + \alpha_0)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}$. We recall the statement of the theorem.

Proposition D.1.

1) $\mathsf{Minimal}(0,k,1)$ *and* $\mathsf{Minimal}(1,k,0)$ *are interreducible.*
2) $\mathsf{Minimal}(1,0,0)$ *reduces to* $\mathsf{Minimal}(1,1,1)$.
3) $\mathsf{Minimal}(1,1,1)$ *reduces to the Minimality Problem for a recurrence of the form*

$$u_n = \frac{\beta_1 n + \beta_0}{n + \alpha} u_{n-1} + \frac{\gamma_1 n + \gamma_0}{n + \alpha} u_{n-2},$$

where the coefficients are elements of $\mathbb{Q}(n)$, $\beta_1 > 0$, and $|\gamma_1| = \beta_1$. The characteristic roots associated to the recurrence are $(\beta_1 \pm \sqrt{\beta_1^2 + 4\gamma_1})/2$. If $\beta_1^2 + 4\gamma_1 = 0$, then we have a further reduction to the Minimality Problem for solutions to

$$u_n = \frac{2n + \beta_0}{n + \alpha} u_{n-1} - \frac{n + \gamma_0}{n + \alpha} u_{n-2},$$

where the coefficients are in $\mathbb{Q}(n)$ and the recurrence has a single repeated characteristic root 1.

4) $\mathsf{Minimal}(1, 0, 1)$ *reduces to the Minimality Problem for solutions to a recurrence of the form*

$$u_n = \frac{\beta_0}{n + \alpha} u_{n-1} + \frac{n + \gamma_0}{n + \alpha} u_{n-2},$$

where $\alpha, \gamma_0 \in \mathbb{Q}$ and $\beta_0 \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$. The characteristic roots associated to the recurrence are ± 1.

Proof of Proposition 5.4.

1) This result follows immediately from the equivalence transformations between (2) and (4).
2) Division by α_1 normalises the recurrence in the following way:

$$(n + \alpha) u_n = \beta u_{n-1} + \gamma u_{n-2}. \tag{16}$$

We shall assume that $\alpha := \alpha_0/\alpha_1 > 1$ (this can be achieved by shifting as appropriate). Suppose that $\langle u_n \rangle_n$ is a solution of the normalised recurrence. We use the reduction argument in Section 2-D to obtain a second recurrence. We find that $(2n + \alpha)(2n + \alpha - 1) u_{2n}$ is equal to

$$(\beta^2 + \gamma(4n + 2\alpha - 3)) u_{2n-2} - \gamma^2 u_{2n-4}. \tag{17}$$

This defines a second-order recurrence with solutions $\langle v_n \rangle_{n=0}^{\infty}$ given by $v_n := u_{2n}$. The mapping $n \mapsto 2n$ establishes a one-to-one correspondence between the solutions of recurrences (16) and (17) and we claim this correspondence preserves minimality. In order to prove this claim we show that linear independence and the asymptotic equalities are preserved. For linear independence one direction is trivial: if $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly dependent solutions of (16), then $\langle u_{2n} \rangle_n$ and $\langle v_{2n} \rangle_n$ are linearly dependent solutions of (17). For the converse, suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly independent. Assume, for a contradiction, that there exists an $\ell \in \mathbb{R}$ such that $u_{2n} = \ell v_{2n}$ for each n. We study the sequence $\langle u_n - \ell v_n \rangle_{n=-1}^{\infty}$. By assumption $0 \neq u_{-1} - \ell v_{-1}$ and $0 = u_0 - \ell v_0$. Using (16) we then compute

$$u_1 - \ell v_1 = \frac{\gamma}{1 + \alpha}(u_{-1} - \ell v_{-1}) \quad \text{and}$$
$$u_2 - \ell v_2 = \frac{\beta}{2 + \alpha}(u_1 - \ell v_1) \neq 0,$$

a contradiction to our assumption.

We turn our attention to minimality. Suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are minimal and dominant solutions of (16), respectively. Then $\lim_{n\to\infty} u_n/v_n = \lim_{n\to\infty} u_{2n}/v_{2n} = 0$. Since $\langle u_{2n} \rangle_n$ and $\langle v_{2n} \rangle_n$ are linearly independent by the above, $\langle u_{2n} \rangle_n$ is necessarily a minimal solution of (17). Conversely, assume that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly independent solutions such that $\langle u_{2n} \rangle_n$ is a minimal solution of (17) (recall that the existence of minimal solutions is decidable for each recurrence). Since $\langle v_{2n} \rangle_n$ is linearly independent of $\langle u_{2n} \rangle_n$, $\lim_{n\to\infty} u_{2n}/v_{2n} = 0$. Since recurrence (16) must also admit minimal solutions, one easily deduces that $\langle u_n \rangle_n$ is likewise minimal.

Notice that the Minimality Problem for $\langle v_n \rangle_n$ is an instance of $\mathsf{Minimal}(2, 1, 0)$ where the polynomial g_3 has rational roots. The equivalence transformations between (2) and (4) give the reduction to $\mathsf{Minimal}(1, 1, 1)$ under this assumption.
3) A solution sequence $\langle u_n \rangle_n$ satisfies a normalised recurrence of the form

$$u_n = \frac{\beta_1 n + \beta_0}{n + \alpha} u_{n-1} + \frac{\gamma_1 n + \gamma_0}{n + \alpha} u_{n-2}.$$

If $\beta_1 = |\gamma_1|$ then we are done. If not, consider the sequence $\langle v_n \rangle_n$ with terms given by $v_n := (\operatorname{sign}(\gamma_1)\beta_1/\gamma_1)^n u_n$. Not only is it evident that $\langle v_n \rangle_n$ satisfies a recurrence of the desired form, but the sequence $\langle v_n \rangle_n$ is also a minimal solution if and only if $\langle u_n \rangle_n$ is a minimal solution.

Assume now that $\beta_1^2 + 4\gamma_1 = 0$ in (9). As $|\gamma_1| = |\beta_1|$, it follows immediately that $\beta_1 = 4 = -\gamma_1$. Now the sequence $\langle (1/2)^n u_n \rangle_n$ satisfies a recurrence of the form (10) and minimality is clearly preserved by this transformation.
4) In this case the recurrence admits minimal solutions if and only if $\gamma_1 \alpha_1 > 0$ (compare to Lemma 5.6). This follows by an application of Theorem 2.8 as the standard normalisation (4) has $\kappa_n = (\gamma_1 n + \gamma_0)(\alpha_1(n-1) + \alpha_0)/\beta_0^2$. The reduction to (11) follows by considering the sequence $\langle (\operatorname{sign}(\beta_0)\sqrt{\alpha_1/\gamma_1})^n u_n \rangle_n$. ☐

APPENDIX E
PROOF OF THEOREM 5.7 COMPLETED

Let us first prove Lemma 5.6.

Proof of Lemma 5.6. First, it is useful to normalise the recurrences using the normalisation in (4) so that one can determine whether a minimal solution exists using the criteria in Theorem 2.8 and Theorem 2.5. Under our assumptions this normalisation does not change the signature of the recurrence. Second, recall that the characteristic roots of a recurrence are real if and only if $\beta_1^2 + 4\gamma_1 \geq 0$. Further, the characteristic roots are distinct if and only if $\beta_1^2 + 4\gamma_1 \neq 0$.

1) Regardless of whether $\gamma_1 = 0$ or not, when the recurrence has signature $(-, +)$ it is clear that $\lim_{n\to\infty} \kappa_n = 0$. When the recurrence has signature $(+, +)$ it is clear that $\sum_{n=2}^{\infty} 1/(n\kappa_n) = \infty$ and so the associated Stern–Stolz series diverges to ∞. These conditions are sufficient to prove the statement.
2) First, let us consider recurrence (9) under the assumption that $\beta_1^2 + 4\gamma \neq 0$. When (9) has signature $(-, +)$ there are minimal solutions if and only if $1 + 4\gamma_1/\beta_1^2 > 0$ if and only if λ, $\mu \in \mathbb{R}$. When (9) has signature $(+, +)$ it is clear that $\lim_{n\to\infty} \kappa_n = \gamma_1/\beta_1^2 > 0$ and so always admits minimal solutions. Finally, recurrence (11) also

always admits minimal solutions: here the recurrence has signature $(+,+)$ and the Stern–Stolz series diverges to ∞ since $\sum_{n=2}^{\infty} 1/\sqrt{\kappa_n} = \infty$.

3) The normalisation of (10) is of the form $w_n = w_{n-1} + \kappa_n w_{n-2}$ with

$$\kappa_n = -\frac{1}{4} - \frac{\alpha - \beta_0 + \gamma_0}{4n} - \frac{\varepsilon}{16n^2} + \mathscr{O}(1/n^3)$$

and $\varepsilon = 4\beta_0(\beta_0 - \alpha - \gamma_0) + 4\alpha(1+\gamma_0) - \beta_0(\beta_0+2)$. There are two cases to consider. If $\beta_0 - \alpha - \gamma_0 \neq 0$ then the recurrence admits minimal solutions if and only if $\beta_0 - \alpha - \gamma_0 > 0$. Otherwise $\beta_0 = \alpha + \gamma_0$, in which case κ_n simplifies as follows:

$$\kappa_n := -\frac{1}{4} - \frac{-(\alpha-\gamma)(\alpha-\gamma-2)}{16n^2} + \mathscr{O}(1/n^3).$$

Since $-x(x-2) \leq 1$ for each $x \in \mathbb{R}$, by Theorem 2.8, we deduce that this subcase always admits minimal solutions. ☐

We move on to complete the proof of Theorem 5.7(1)&(2)&(4) in order.

Proof of Theorem 5.7(1). If $\langle u_n \rangle_n$ is a minimal solution to recurrence $u_n = (\beta_1 n + \beta_0)u_{n-1} + \gamma_0 u_{n-2}$ then, by Theorem 2.5, $-u_0/u_{-1}$ is equal to

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{\gamma_0}{\beta_1 n + \beta_0} = \beta_0 \frac{{}_0F_1(;\beta_0/\beta_1;\gamma_0/\beta_1^2)}{{}_0F_1(;\beta_0/\beta_1+1;\gamma_0/\beta_1^2)} - \beta_0$$

(see [37, §VI.4]). Hence the Minimality Problem for the above recurrence reduces to checking the equality

$$(\beta_0 u_{-1} - u_0)\, {}_0F_1\left(;\frac{\beta_0}{\beta_1}+1;\frac{\gamma_0}{\beta_1^2}\right) = u_{-1}\beta_0\, {}_0F_1\left(;\frac{\beta_0}{\beta_1};\frac{\gamma_0}{\beta_1^2}\right).$$

The *Bessel functions of the first kind*, sometimes called *cylinder functions*, $J_s(z)$ are a family of functions that solve Bessel's differential equation [1], [3], [8]. For $z, s \in \mathbb{C}$ the function $J_s(z)$ is defined by the hypergeometric series [1, Equation 9.1.69]

$$J_s(z) := \frac{1}{\Gamma(s+1)}\left(\frac{z}{2}\right)^s {}_0F_1(;s+1;-z^2/4).$$

We obtain the principal branch of $J_s(z)$ by assigning $(z/2)^s$ its principal value. When $\mathrm{Re}(s) > -1/2$ we have the following integral representation [3, Equation 4.7.5],

$$J_s(z) = \frac{1}{\sqrt{\pi}\Gamma(s+1/2)}\left(\frac{z}{2}\right)^s \int_{-1}^{1} \mathrm{e}^{\mathrm{i}zt}(1-t^2)^{s-1/2}\,dt.$$

Hence for $\mathrm{Re}(s) > -1/2$, we have the following integral representation

$${}_0F_1(;s+1;z) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+1/2)} \int_{-1}^{1} \mathrm{e}^{-2\sqrt{z}t}(1-t^2)^{s-1/2}\,dt.$$

Let us return to minimal solutions of the aforementioned recurrence relation. By substitution and linearity of the integral, we see that $\mathsf{Minimal}(0,1,0)$ reduces to checking whether the following integral

$$\int_{-1}^{1} \mathrm{e}^{\frac{-2\sqrt{\gamma_0}}{\beta_1}t}(1-t^2)^{\beta_0/\beta_1-3/2}\Big(\tfrac{2(\beta_0 u_{-1}-u_0)}{2\beta_0-1}(1-t^2)-1\Big)\,dt$$

vanishes. The integral in question is an exponential period, as the values β_i and γ_0 are rational numbers. To ensure that the integral converges absolutely note that we can shift the recurrence so that $\beta_0/\beta_1 > 3/2$. ☐

Proof of Theorem 5.7(2). Consider an instance of $\mathsf{Minimal}(0,1,1)$: without loss of generality it is of the form $u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}$. By shifting the sequence appropriately, we may assume that $\frac{\beta_0}{\beta_1} + \frac{\gamma_1}{\beta_1^2}$ is positive. Let $\langle u_n \rangle_n$ be a minimal solution. Then $-u_0/u_{-1}$ is equal to

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{\gamma_1 n + \gamma_0}{\beta_1 n + \beta_0} = \frac{(\beta_0 + \frac{\gamma_1}{\beta_1})\, {}_1F_1(\frac{\gamma_0}{\gamma_1};\frac{\beta_0}{\beta_1}+\frac{\gamma_1}{\beta_1^2};\frac{\gamma_1}{\beta_1^2})}{{}_1F_1(\frac{\gamma_0}{\gamma_1}+1;\frac{\beta_0}{\beta_1}+\frac{\gamma_1}{\beta_1^2}+1;\frac{\gamma_1}{\beta_1^2})} - \beta_0.$$

Here the value on the right-hand side is given in [37, §VI.4].

Let $a = \gamma_0/\gamma_1$, $b = \beta_0/\beta_1 + \gamma_1/\beta_1^2$, and $x = \gamma_1/\beta_1^2$, then $\mathsf{Minimal}(0,1,1)$ reduces to checking the equality

$$\frac{{}_1F_1(a;b;x)}{{}_1F_1(a+1;b+1;x)} = \frac{(u_{-1}\beta_0 - u_0)}{u_{-1}(\beta_0 + \frac{\gamma_1}{\beta_1})}.$$

Notice here that $a > 0$ and $b > 0$ by assumption. Applying the transformation ${}_1F_1(a;b;x) = \mathrm{e}^x\, {}_1F_1(b-a;b;-x)$ (cf. [1, 13.1.27]), we have that if $b-a$ is zero or a negative integer, the values of the associated finite series are elements of $\mathbb{Q}(a,b,x)$. As these parameters are rational, the equality is plain to check. We thus assume that $b-a$ is not zero or a negative integer.

Proceeding as in the proof of Theorem 5.7(3), it can be shown that the sequence $\langle P_n(x) \rangle_{n=0}^{\infty}$ with $P_n = {}_1F_1(a+n;b+2n;x)$ if n is even, $P_n = {}_1F_1(a+n+1;c+2n+1;x)$ if n is odd, satisfies

$$\frac{P_0(x)}{P_1(x)} = s_1 \circ \cdots \circ s_n\left(\frac{P_n(x)}{P_{n+1}(x)}\right),$$

where $\langle s_n \rangle_{n=1}^{\infty}$ is a sequence of linear fractional transformations given by $s_n(w) = d_n x/(1+w)$, where

$$d_n = \begin{cases} -\dfrac{b-a+n}{(b+2n)(b+2n+1)} & \text{if } n \text{ is odd, and} \\ \dfrac{a+n}{(b+2n-1)(b+2n)} & \text{if } n \text{ is even} \end{cases}$$

(see [37, §VI.2.2]). As $d_n \neq 0$ under our assumptions on a and b, the composition of the linear fractional transformations is an invertible. Thus

$$\frac{P_0(x)}{P_1(x)} = \frac{X_n P_{n+1}(x) + Y_n P_n(x)}{Z_n P_{n+1}(x) + W_n P_n(x)},$$

with $X_n W_n - Y_n Z_n \neq 0$ for all $n \geq 1$. We recall that ${}_1F_1(a;b;x)$ admits the integral representation

$${}_1F_1(a;b;x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1}\mathrm{e}^{xt}\,dt$$

whenever $\mathrm{Re}(b) > \mathrm{Re}(a) > 0$ [37, Appendix 4.4] or [11, §6.5]. One can then complete the proof in a manner analogous to the proof of Theorem 5.7(3). □

It remains to prove Theorem 5.7(4). We first deal with a simple case that turns out to be decidable.

Lemma E.1. *Let $\langle u_n \rangle_n$ be a non-trivial solution to* (10) *with $\beta_0 = \alpha + \gamma_0$. If $\alpha \le \gamma_0 + 1$ then $\langle u_n \rangle_n$ is minimal if and only if $u_0/u_{-1} = 1$. If $\alpha > \gamma_0 + 1$ then $\langle u_n \rangle_n$ is minimal if and only if $u_0/u_{-1} = (\gamma_0 + 1)/\alpha$.*

Proof. If $\beta_0 = \alpha + \gamma_0$, then the constant sequence $\langle 1 \rangle_n$ is a solution to the recurrence by inspection. Hence $\langle 1 \rangle_n$ and $\langle B_n \rangle_n$ defined by $B_{-1} = 0$, $B_0 = 1$ are linearly independent solutions, and by Lemma 2.10

$$B_n = \sum_{k=0}^{n} \prod_{m=1}^{k} \frac{m + \gamma_0}{m + \alpha} = \sum_{k=0}^{n} \frac{(\gamma_0 + 1)_k}{(\alpha + 1)_k}.$$

By a straightforward application of Stirling's approximation, $(\gamma_0+1)_n/(\alpha+1)_n \sim n^{\gamma_0-\alpha}$ as $n \to \infty$. Hence if $\gamma_0 - \alpha \ge -1$ the series diverges (by comparison to the harmonic series) from which we deduce that $\langle 1 \rangle_n$ is minimal. If $\gamma_0 - \alpha < -1$, then $\lim_{n\to\infty} B_n$ converges to the value

$$\sum_{k=0}^{\infty} \frac{(\gamma_0 + 1)_k (1)_k}{(\alpha + 1)_k} \frac{1}{k!} = {}_2F_1(\gamma_0 + 1, 1; \alpha + 1; 1)$$
$$= \frac{\Gamma(\alpha + 1)\Gamma(\alpha - \gamma_0 - 1)}{\Gamma(\alpha - \gamma_0)\Gamma(\alpha)} = \frac{\alpha}{\alpha - \gamma_0 - 1}.$$

In the second equality we use [3, Thm. 2.2.2]. It follows that $\langle u_n \rangle_n = \frac{\alpha}{\alpha-\gamma_0-1} \langle 1 \rangle_n - \langle B_n \rangle_n$ is a minimal solution, and we may compute $u_0/u_{-1} = (\gamma_0 + 1)/\alpha$. □

Proof of Theorem 5.7(4). The case when $\beta_0 = \alpha + \gamma_0$ is decidable by the above lemma, so we consider the case $\beta_0 > \alpha + \gamma_0$; otherwise the recurrence admits no minimal solutions by Lemma 5.6.

The function $U(a, b, x)$, the *confluent hypergeometric function of the second kind*, is defined for all $a, b, x \in \mathbb{C}$ with $\mathrm{Re}(a), \mathrm{Re}(x) > 0$ as

$$U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-xt} t^{a-1} (1 + t)^{b-a-1} \, dt.$$

As noted by Temme in [53], the sequence $\langle u_n \rangle_{n=-1}^{\infty}$ given by $u_{-1} = \frac{1}{a-1} U(a - 1, b, x)$, $u_n := (a)_n U(a + n, b, x)$ is a minimal solution of the recurrence

$$u_n = \frac{2n + x + 2a - b - 2}{n + a - b} u_{n-1} - \frac{n + a - 2}{n + a - b} u_{n-2} \qquad (18)$$

(assuming that $a \ne 1$ and $a - b$ is not a negative integer). Notice that the recurrence holds also for $n = 1$ because

$$(2a+x-b)U(a, b, z) - U(a-1, b, z) = a(1+a-b)U(a+1, b, c).$$

When one substitutes the values $a = \gamma_0 + 2$, $b = \gamma_0 + 2 - \alpha$, and $x = \beta_0 - \gamma_0 - \alpha$ into (10) one obtains recurrence (18). Subject to an initial shift of the sequence, we may assume that have $a > 2$. We also have $x > 0$ by assumption (shifting the sequence has no effect on x). Hence, a minimal solution to (10) satisfies $u_0/u_{-1} = (a - 1)U(a, b, x)/U(a - 1, b, x)$.

We may apply the integral representation for U immediately. Since the parameters involved are rational numbers, the integrals obtained are exponential periods, and the claim follows. □

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