On minimality and positivity for second-order holonomic sequences

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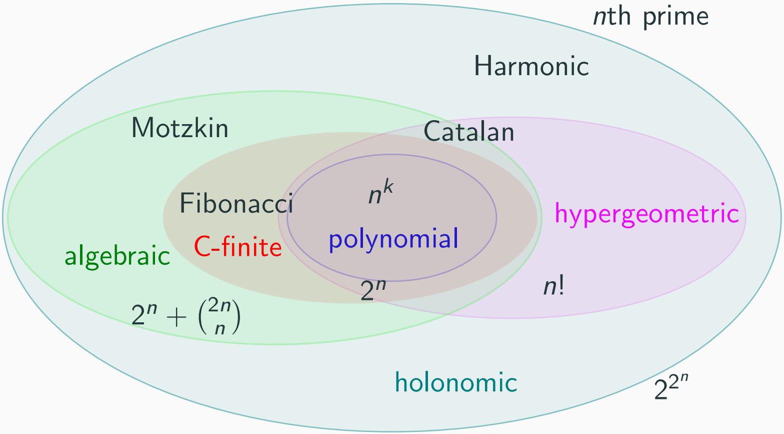
classes of recurrence sequences[1]



[1] see Kauers and Paule, 2011

second-order holonomic sequences

Definition
A second-order holonomic sequence $\langle u_n \rangle_n$ satisfies a polynomial recurrence relation

$$p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2}$$

for each $n \in \mathbb{N}$. Here $p_1, p_2, p_3 \in \mathbb{Q}[n]$ and $p_1(n), p_3(n) \neq 0$.

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Examples

- Fibonacci's sequence satisfies $u_n = u_{n-1} + u_{n-2}$.
- Apéry's sequence $u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies

$$n^3 u_n = (34n^3 - 51n^2 + 27n - 5)u_{n-1} - (n-1)^3 u_{n-2}.$$

two decision problems

Definition
A non-trivial holonomic sequence $\langle u_n \rangle_n$

- is minimal if, for every linearly independent solution $\langle v_n \rangle_n$ to the same recurrence, $\lim_{n\to\infty} u_n/v_n = 0$;
- is positive if $u_n \geq 0$ for each n.

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$\zeta(3) := \sum_{k=1}^{\infty} k^{-3}$ is irrational as $\zeta(3)A_n - B_n \to 0$ quickly.

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Since $\lim_{n\to\infty} \zeta(3)A_n - B_n = 0$, $\langle u_n \rangle_n$ is minimal precisely when

$$u_1/u_0 = (5\zeta(3) - 6)/\zeta(3).$$

Theorem
Given a second-order polynomial recurrence relation,

There is a subinterval $P \subseteq \mathbb{R}$ such that a non-trivial solution $\langle u_n \rangle_{n=-1}^{\infty}$ is positive if and only if $u_0/u_{-1} \in P$.

We can decide positivity except when u_0/u_{-1} coincides with an endpoint of P.

For second-order holonomic sequences, the Positivity Problem Turing-reduces to the Minimality Problem.

approximations for $\pi/4$

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}}$$

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$$\frac{\pi}{4} = \cfrac{1}{1 + \underset{j=2}{\overset{\infty}{\mathrm{K}}} \frac{(2j-3)^2}{2}}$$

We employ Gauss' Kettenbruch notation for brevity!

approximations for $\pi/4$

$$\cfrac{1}{1 + \underset{j=2}{\overset{\infty}{\mathrm{K}}} \cfrac{(2j-3)^2}{2}}$$

Recursively define $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$ with
$A_0 = 0, A_1 = 1, B_0 = 1, B_1 = 1$, and for $n \geq 2$

$$A_n = 2A_{n-1} + (2n-3)^2 A_{n-2}; \text{ and}$$

$$B_n = 2B_{n-1} + (2n-3)^2 B_{n-2}.$$

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(Here $(2n-1)!! := (2n-1)(2n-3)\cdots 3 \cdot 1$.)

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and so in the limit as $n \to \infty$

$$\frac{1}{1 + \mathop{\mathrm{K}}_{j=2}^{\infty} \frac{(2j-3)^2}{2}} = \arctan(1) = \frac{\pi}{4}.$$

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- $\langle w_n \rangle_n$ is positive if and only if $\langle u_n \rangle_n$ is positive.

Pincherle's Theorem

Theorem (Pincherle, 1894)
The recurrence $w_n = w_{n-1} + \kappa_n w_{n-2}$ admits a minimal solution if and only if $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ converges.

Further, if $\langle w_n \rangle_{n=-1}^{\infty}$ is a minimal solution then $-w_0/w_{-1} = \mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$.

Worpitzky's Theorem

Theorem (Generalised Worpitzky[2])
$\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ *converges if and only if, either*

- $0 > \lim_{n\to\infty} \kappa_n > -1/4$, *or*
- $\lim_{n\to\infty} \kappa_n = -1/4$ *and eventually* $\kappa_n \geq -1/4 - 1/(4n)^2 - 1/(4n \log n)^2$.

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If one of the above holds then[3]

- $\langle A_n/B_n \rangle_n$ strictly decreases and converges to a finite value
- $B_n > 0$ for each $n \in \mathbb{N}$.

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Hence the non-trivial sequence $\langle w_n \rangle_n$ is positive if and only if $-w_0/w_{-1} \leq \mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$.

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Task: detect when $-w_0/w_{-1} < \mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$.
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- If $\langle w_n \rangle_n$ is a non-trivial solution then, for each n

$$-w_n/w_{n-1} \in \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}.$$

- If $\langle w_n \rangle_n$ is positive then $-w_n/w_{n-1} < 0$ for each n.
- $$-w_n/w_{n-1} = -1 + \frac{\kappa_n}{-w_{n-1}/w_{n-2}}.$$

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polynomial continued fractions

For $\langle w_n \rangle_n$ a solution to $w_n = w_{n-1} + \kappa_n w_{n-2}$,

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For $\langle u_n \rangle_n$ a holonomic sequence,

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$$\zeta(3) = \cfrac{6}{5 + \operatorname*{K}_{n=1}^{\infty} \frac{-n^6}{34n^3 + 51n^2 + 27^n + 5}}$$

Definition (Kontsevich and Zagier, 2001)
A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of algebraic functions with algebraic coefficients over domains in $\mathbb{R}^k$ given by polynomial inequalities with algebraic coefficients

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equality between periods

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It is decidable whether two periods are equal.

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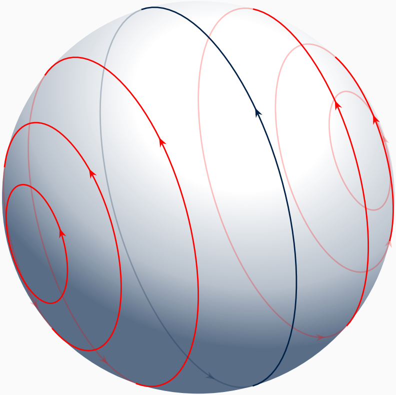
Proof (Idea).
Study the convergence properties of the generating function whose coefficient sequence is minimal. □

Thank you!

intuition

Behaviour of $\langle w_n / w_{n-1} \rangle_n$ when $\langle w_n \rangle_n$ is minimal.



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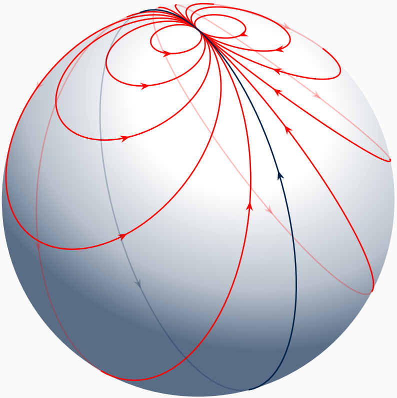

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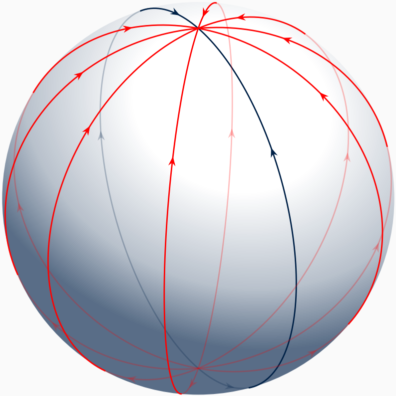

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references i

- L. Jacobsen and D. R. Masson. “On the convergence of limit periodic continued fractions $K(a_n/1)$, where $a_n \to -\frac{1}{4}$. III”. In: *Constr. Approx.* 6.4 (1990), 363–374.
- Manuel Kauers and Peter Paule. *The concrete tetrahedron*. Texts and Monographs in Symbolic Computation. Symbolic sums, recurrence equations, generating functions, asymptotic estimates. SpringerWienNewYork, Vienna, 2011, pp. x+203.
- Maxim Kontsevich and Don Zagier. “Periods”. In: *Mathematics unlimited—2001 and beyond*. Springer, Berlin, 2001, 771–808.

Lisa Lorentzen and Haakon Waadeland. *Continued fractions. Vol. 1*. Second. Vol. 1. Atlantis Studies in Mathematics for Engineering and Science. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008, pp. xii+308.

Salvatore Pincherle. "Delle Funzioni ipergeometriche, e di varie questioni ad esse attinenti". In: *Giorn. Mat. Battaglini* 32 (1894), 209–291.